On vector bundles over algebraic and arithmetic curves

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Introduction

This text deals with vector bundles over a curve X. It consists of two independent parts: In the first two chapters, X is a smooth projective curve with a marked point P over an algebraically closed field, and the vector bundles are endowed with parabolic structures, i. e. with a chain of vector subspaces in the fibre over P (the quasiparabolic structure) and some real numbers (the weights). In the third chapter, X is an arithmetic curve, i. e. the set of places of a number field, and the vector bundles are metrized bundles in the sense of Arakelov geometry.

There is some analogy between parabolic and Arkelov vector bundles: The latter are given by (archimedean) norms at the infinite places, and the former can be described by non-archimedean norms at the marked point(s) like in [HJS98]. However, the questions, methods or results of the first two chapters are not analogous to those of the third chapter.

The subject of the first two chapters is a conjecture of H. Boden and Y. Hu about their desingularisation of the moduli scheme of semistable parabolic bundles. Recall that the notion of (Mumford) semistability depends on the weights. Boden and Hu observed that a slight variation of the weights leads to a desingularisation of the moduli scheme, and they conjectured that one can always obtain a *small* resolution this way.

In the present text, it is proved that this conjecture holds for bundles of rank up to eight, but not for rank nine and beyond. This is a consequence of theorem 2.2.5 which states that the Boden-Hu desingularising map is a Zariski-locally trivial fibration over each Jordan-Hölder stratum of the singular moduli scheme and gives a quite precise description of the typical fibre.

The starting point for the proof of theorem 2.2.5 is the observation that the fibres in question parameterize multiple extensions of quasiparabolic bundles. By an extension of several given bundles, we mean a bundle E together with a chain of subbundles such that the resulting subquotients of E are isomorphic to the given bundles. The first chapter is devoted to the study of such extensions; its main results are the construction and description of fine moduli schemes of extensions.

Theorem 2.2.5 will then follow from an intimate relation between the fibres of the Boden-Hu map and moduli schemes of extensions. (For extensions of two bundles, this relation has already been noted and exploited by Boden and Hu.)

The dimension formulas of theorem 2.2.5 reduce the Boden-Hu conjecture to an elementary question about weights, not involving the curve X any more. This elementary problem is studied in the last three sections of chapter two.

The background of the independent chapter three can be described by the following task: Prove the existence of Arakelov vector bundles $\mathcal{E}$ over the arithmetic curve X without nonzero global sections and with the degree of $\mathcal{E}$ as large as possible. If X is the set of places of $\mathbb{Q}$, then this is the problem of lattice sphere packing. The analogous question for vector bundles over algebraic curves has a simple answer: The degree is maximal if the Euler characteristic vanishes, so the maximal slope is $g - 1$ by Riemann-Roch.

G. Faltings has proved that for each semistable vector bundle E over an algebraic curve, there is another vector bundle F such that $E \otimes F$ has no global sections and slope $g - 1$. (See [Fal93] and [Fal96] where this result is interpreted in terms of theta functions and used to construct moduli schemes of vector bundles without appeal to Mumford's Geometric Invariant Theory.) The main result 3.4.6 of chapter three can be seen as an arithmetic analogue of this theorem; it states that for each semistable Arakelov vector bundle $\mathcal{E}$, there is another Arakelov bundle $\mathcal{F}$ such that $\mathcal{E} \otimes \mathcal{F}$ has no global sections and slope larger than a certain bound.

The proof of 3.4.6 is inspired by that of the Minkowski-Hlawka existence theorem for sphere packings: It is not constructive and uses integration over a space of Arakelov bundles (with respect to some Tamagawa measure). With an adelic version of Siegel's mean value formula, the average number of nonzero global sections can be computed; if it is less than one, at least one of the bundles has no global sections.

In contrast to its algebraic counterpart $g - 1$, the resulting bound on the slope of $\mathcal{E} \otimes \mathcal{F}$ depends on the rank of $\mathcal{F}$. It is best if the rank of $\mathcal{E}$ is one, so considering tensor products this way does not produce better sphere packings than the original Minkowski-Hlawka theorem.

I would like to thank my thesis adviser G. Faltings for his suggestions, his support and encouragement. Especially the third chapter is based on his ideas. I also had many fruitful discussions with my colleagues in Bonn, especially with Jochen Heinloth. Markus Rosellen has drawn my attention to the paper of Boden and Hu. The work was supported by a grant of the Max-Planck-Institut in Bonn and by the excellent working conditions there. For some time, the author was supported by the university of Bonn.

Chapter 1

Multiple extensions of quasiparabolic bundles

1.1 Quasiparabolic bundles

This section recalls the notion of (families of) vector bundles with quasiparabolic structures that was introduced in [MeSe80]. The main purpose is to fix notation and to collect some basic facts.

Once and for all, we fix a smooth connected projective curve X of genus g over an algebraically closed field k and a closed point $P \in X(k)$. Furthermore, we fix a positive integer N which will later become the number of weights.

We will always use the following conventions: A vector bundle E over a scheme S is a locally free coherent sheaf. A subbundle E' of E is a coherent subsheaf of E that is locally a direct summand. We denote by $\mathrm{Tot}(E) = \mathrm{Tot}_S(E)$ the total space of E; this is a scheme over S.

Definition 1.1.1 *Let S be a scheme over k, considered as a parameter space.*

i) *A* quasiparabolic bundle *E over $X \times_k S$ is a vector bundle $\check{E}$ over $X \times_k S$ together with a filtration of its restriction $\check{E}_P$ to $\{P\} \times S$ by subbundles*

$$\check{E}_P = F_0\check{E}_P \supseteq F_1\check{E}_P \supseteq \ldots \supseteq F_N\check{E}_P = 0.$$

ii) *A* morphism *$\phi : E \to E'$ of quasiparabolic bundles E and E' over $X \times_k S$ is a morphism of vector bundles $\check{\phi} : \check{E} \longrightarrow \check{E}'$ whose restriction $\check{\phi}_P : \check{E}_P \longrightarrow \check{E}'_P$ respects the given filtrations, i. e. satisfies*

$$\check{\phi}_P(F_n\check{E}_P) \subseteq F_n\check{E}'_P$$

for all $n \leq N$.

We have the *pullback* f^*E of the quasiparabolic bundle E along a morphism $f : T \to S$ of k-schemes; it is the quasiparabolic bundle over $X \times_k T$ that consists of the vector bundle $f^*\check{E}$ and the filtration $(f^*F_n\check{E}_P)_{n\leq N}$. In particular, the *fibre* E_s of E over a point s of S is its pullback to the spectrum of the residue field $k(s)$.

Such quasiparabolic bundles have several discrete invariants, namely the *rank* $\mathrm{rk}(E)$, the *underlying degree* $\deg(\check{E})$ and the *multiplicities*

$$m_1 := \mathrm{rk}(\check{E}_P/F_1\check{E}_P)\ ,\ m_2 := \mathrm{rk}(F_1\check{E}_P/F_2\check{E}_P)\ , \ldots,\ m_N := \mathrm{rk}(F_{N-1}\check{E}_P)\ .$$

All these are locally constant functions on S with integer values and will often be fixed. Note that the rank equals the sum of the multiplicities. We collect the discrete invariants in a single notion:

Definition 1.1.2 *A* multiplicity vector m *is a sequence of integers*

$$m = (r, \check{d}, m_1, \dots, m_N)$$

such that all the m_n *are nonnegative and*

$$r = m_1 + m_2 + \dots + m_N$$

holds.

Of course, the multiplicity vector of a quasiparabolic bundle E over $X \times_k S$ (near some point s of S) consists of its rank r, its underlying degree $\check{d}$ and its multiplicities $m_1, \dots, m_N$ (near s).

Remark 1.1.3 Although it may seem to be a bit unusual, we deliberately allow zero multiplicities. This has a consequence for the notion of isomorphism: Two quasiparabolic bundles E and E' cannot be isomorphic if their multiplicity vectors m and m' are different, even if they have the same *nonzero* multiplicities like $m = (1, \check{d}, 1, 0)$ and $m' = (1, \check{d}, 0, 1)$.

Definition 1.1.4 *Let* S *be a* k*-scheme. A collection of quasiparabolic bundles over* $X \times_k S$ *and morphisms*

$$0 \longrightarrow E^1 \xrightarrow{\iota} E \xrightarrow{\pi} E^2 \longrightarrow 0 \tag{1.1}$$

is a (short) exact sequence *if the morphisms of the underlying vector bundles*

$$0 \longrightarrow \check{E}^1 \xrightarrow{\check{\iota}} \check{E} \xrightarrow{\check{\pi}} \check{E}^2 \longrightarrow 0$$

form an exact sequence, and their restrictions to $\{P\} \times S$ *induce an exact sequence*

$$0 \longrightarrow F_n\check{E}^1_P \longrightarrow F_n\check{E}_P \longrightarrow F_n\check{E}^2_P \longrightarrow 0$$

for each $n \leq N$.

If (1.1) is an exact sequence, then the multiplicity vector of E is the sum of the multiplicity vectors of E^1 and E^2.

We say that a quasiparabolic bundle E' is a *subbundle* of a quasiparabolic bundle E if $\check{E}'$ is a subbundle of $\check{E}$ and the condition

$$F_n\check{E}'_P = \check{E}'_P \cap F_n\check{E}_P$$

is satisfied for all $n \leq N$.

If (1.1) is an exact sequence, then ι is an isomorphism onto a subbundle of E. Conversely, if we have a subbundle E' of E, then we can define the (quasiparabolic) *quotient bundle* E/E' by the vector bundle $\check{E}/\check{E}'$ and the filtration

$$(\check{E}/\check{E}')_P = F_0\check{E}_P/F_0\check{E}'_P \supseteq F_1\check{E}_P/F_1\check{E}'_P \supseteq \dots \supseteq F_N\check{E}_P/F_N\check{E}'_P = 0,$$

thus obtaining an exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E/E' \longrightarrow 0.$$

We say that a morphism of quasiparabolic bundles $\phi : E \to E''$ is *surjective* if the induced morphisms of vector bundles

$$\check{\phi} : \check{E} \longrightarrow \check{E}'' \quad \text{and} \quad \check{\phi}_P : F_n\check{E}_P \longrightarrow F_n\check{E}''_P$$

are all surjective. In this case, the kernel of ϕ is a subbundle E' of E, and

$$0 \longrightarrow E' \longrightarrow E \xrightarrow{\phi} E'' \longrightarrow 0$$

is an exact sequence.

Occasionally, it will be useful to work only over open subsets of $X \times_k S$. Here is the obvious definition:

Definition 1.1.5 *Let U be an open subscheme of $X \times_k S$.*

i) If $U \cap (\{P\} \times S)$ is empty, then a quasiparabolic bundle *E over U is just a vector bundle over U. Otherwise, it is a vector bundle $\check{E}$ over U together with a filtration of its restriction $\check{E}_P$ to $U \cap (\{P\} \times S)$ by subbundles*

$$\check{E}_P = F_0\check{E}_P \supseteq F_1\check{E}_P \supseteq \ldots \supseteq F_N\check{E}_P = 0.$$

ii) If $U \cap (\{P\} \times S)$ is empty, then a morphism *$\phi : E \to E'$ of quasiparabolic bundles E and E' over U is just a morphism of vector bundles. Otherwise, it is a morphism of vector bundles $\check{\phi} : \check{E} \longrightarrow \check{E}'$ whose restriction $\check{\phi}_P : \check{E}_P \longrightarrow \check{E}'_P$ respects the given filtrations.*

Note that one can glue quasiparabolic bundles. More precisely, let

$$X \times_k S = U_1 \cup U_2$$

be an open covering and assume given quasiparabolic bundles E_i over U_i together with an isomorphism

$$\phi : E_2\big|_{U_1 \cap U_2} \xrightarrow{\sim} E_1\big|_{U_1 \cap U_2}.$$

Then one can form the quasiparabolic bundle

$$E := E_1 \cup_\phi E_2.$$

Proposition 1.1.6 *Let E and E' be quasiparabolic bundles over $X \times_k S$, and define the $\mathcal{O}_{X \times_k S}$-module sheaf of (local) morphisms $\mathcal{H}om(E, E')$ by*

$$\Gamma(U, \mathcal{H}om(E, E')) := \mathrm{Hom}(E\big|_U, E'\big|_U).$$

i) $\mathcal{H}om(E, E')$ is a vector bundle over $X \times_k S$. As locally constant functions on S, its rank is $r \cdot r'$ and its degree is

$$r\check{d}' - r'\check{d} - \sum_{1 \le n' < n \le N} m_n \cdot m'_{n'}$$

where $(r, \check{d}, m_1, \ldots, m_N)$ and $(r', \check{d}', m'_1, \ldots, m'_N)$ are the multiplicity vectors of E and E', respectively.

ii) $\mathcal{H}om$ commutes with base change. More precisely, the canonical morphism

$$f^*\mathcal{H}om(E,E') \longrightarrow \mathcal{H}om(f^*E, f^*E')$$

is an isomorphism for every morphism of k-schemes $f : T \longrightarrow S$.

iii) If (1.1) is an exact sequence of quasiparabolic bundles, then

$$0 \longrightarrow \mathcal{H}om(E', E^1) \xrightarrow{\iota_*} \mathcal{H}om(E', E) \xrightarrow{\pi_*} \mathcal{H}om(E', E^2) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{H}om(E^2, E') \xrightarrow{\pi^*} \mathcal{H}om(E, E') \xrightarrow{\iota^*} \mathcal{H}om(E^1, E') \longrightarrow 0$$

are also exact.

Proof: All statements are local in S, so we may assume without loss of generality that S is affine. Then $\{P\} \times S$ is affine as well, so all our subbundles are direct summands. Choosing complements gives us an exact sequence

$$0 \to \mathcal{H}om(E,E') \to \mathcal{H}om(\check{E},\check{E}') \to \bigoplus_{1 \le n' < n \le N} \mathcal{H}om(\frac{F_{n-1}\check{E}_P}{F_n\check{E}_P}, \frac{F_{n'-1}\check{E}'_P}{F_{n'}\check{E}'_P}) \to 0 \tag{1.2}$$

of $\mathcal{O}_{X\times_k S}$-module sheaves in which the last term is a vector bundle over $\{P\} \times S$ of rank $\sum_{n'<n} m_n m'_{n'}$. Everything follows from this:

First of all, it implies that $\mathcal{H}om(E,E')$ is coherent over $X \times_k S$ and flat over S as the other terms are.

Secondly, ii follows from it: Of course the middle term commutes with base change, and the last term also does if we choose our complements over T by pulling back the chosen complements over S. Now ii follows from the flatness of the last term.

Thirdly, it immediately implies i if S is the spectrum of a field. The case of general S follows from this using ii and the local criterion for flatness.

Finally, iii is deduced as follows: Let (1.1) be an exact sequence of quasiparabolic bundles. (1.2) is functorial with respect to ι and π if we choose the required complements for E^1 and E^2 first and map the former, lift the latter to E to get the complements there. Now the second and the third term of (1.2) are exact functors of both variables, so the same holds for the first term by the 3×3-lemma for coherent sheaves. □

Note that the degree of the vector bundle $\mathcal{H}om(E,E')$ depends only on the multiplicity vectors of E and E'. In later sections, especially the antisymmetric part of this function will be of interest, so we introduce a symbol for it:

Definition 1.1.7 *Let $m = (r, \check{d}, m_1, \ldots, m_N)$ and $m' = (r', \check{d}', m'_1, \ldots, m'_N)$ be multiplicity vectors. The integer valued, antisymmetric function Δ of m and m' is defined by*

$$\Delta(m,m') = 2r\check{d}' + \sum_{n<n'} m_n m'_{n'} - 2r'\check{d} - \sum_{n'<n} m_n m'_{n'}.$$

Corollary 1.1.8 *If $m = (r, \check{d}, m_1, \ldots, m_N)$ and $m' = (r', \check{d}', m'_1, \ldots, m'_N)$ are the multiplicity vectors of quasiparabolic bundles E and E' over $X \times_k S$, then the degree of the sheaf of morphisms from E to E' is*

$$\deg(\mathcal{H}om(E,E')) = -\frac{rr'}{2} + \sum_{n=1}^{N} \frac{m_n m'_n}{2} + \frac{1}{2}\Delta(m,m').$$

Proposition 1.1.6 also has consequences for the behavior of Hom under base change. Remember that a coherent sheaf $\mathcal{F}$ over $X \times_k S$ is called *cohomologically flat* if $p_*\mathcal{F}$ and $\mathrm{R}^1 p_*\mathcal{F}$ are vector bundles over S and commute with any base change $f : T \to S$, the latter meaning that the natural morphism

$$f^*\mathrm{R}^i p_*\mathcal{F} \longrightarrow \mathrm{R}^i p_{T,*} f^*\mathcal{F}$$

is an isomorphism for $i = 0, 1$. Here $p_T : X \times_k T \longrightarrow T$ denotes the pullback of $p : X \times_k S \longrightarrow S$.

Recall that E is *simple* if the canonical map from the residue field $k(s)$ to $\mathrm{End}(E_s)$ is an isomorphism for every point s of S. Part ii of the following corollary states in particular that $\mathcal{E}nd(E) := \mathcal{H}om(E, E)$ is cohomologically flat if E is simple.

Corollary 1.1.9 *Let E and E' be quasiparabolic bundles over $X \times_k S$.*

i) We say that $\mathrm{Hom}(E, E')$ vanishes fibrewise *if* $\mathrm{Hom}(E_s, E'_s) = 0$ *holds for all points s of S. If this is the case, then $\mathcal{H}om(E, E')$ is cohomologically flat and $p_*\mathcal{H}om(E, E')$ vanishes.*

ii) If there is a morphism $\phi : E \to E'$ such that

$$\cdot\phi_s : k(s) \longrightarrow \mathrm{Hom}(E_s, E'_s)$$

is an isomorphism for all points s of S, then $\mathcal{H}om(E, E')$ is cohomologically flat and

$$\cdot\phi : \mathcal{O}_S \longrightarrow p_*\mathcal{H}om(E, E')$$

is an isomorphism, too.

iii) If the dimension of $\mathrm{Hom}(E_s, E'_s)$ *is at most one for each point s of S, then there is a unique closed subscheme Z of S with the following universal property:*

Any k-morphism $f : T \to S$ factors through Z if and only if $\mathcal{H}om(f^*E, f^*E')$ is cohomologically flat and $p_{T,*}\mathcal{H}om(f^*E, f^*E')$ is a line bundle.

Proof: We use a main result of [EGA III]; applied to $\mathcal{H}om(E, E')$, it states that locally in S, there is a complex of length one consisting of vector bundles over S

$$\mathcal{F}^0 \xrightarrow{\delta} \mathcal{F}^1$$

which represents the cohomology in the sense that for every base change $f : T \to S$, the direct images

$$p_{T,*}f^*\mathcal{H}om(E, E') \quad \text{and} \quad \mathrm{R}^1 p_{T,*} f^*\mathcal{H}om(E, E')$$

are the kernel and the cokernel of its pullback

$$f^*\mathcal{F}^0 \xrightarrow{\delta_T} f^*\mathcal{F}^1.$$

All claims are local with respect to S, so we may assume that S is the spectrum of a k-algebra A and that the complex of finitely generated free A-modules

$$M^0 \xrightarrow{\delta} M^1$$

represents the cohomology of $\mathcal{H}om(E, E')$.

The hypothesis of i means that δ is injective modulo every prime ideal of A. This implies that δ is an isomorphism onto a direct summand: For this step, we may assume that A is local with maximal ideal $\mathfrak{m}$, and then injectivity modulo $\mathfrak{m}$ means that the matrix of δ has maximal rank modulo $\mathfrak{m}$, so it contains a maximal quadratic submatrix which is invertible. Hence M^0 is indeed a direct summand in M^1, and i follows.

Under the assumptions of ii, the multiplication with ϕ is a linear map

$$A \longrightarrow M^0$$

which is injective modulo all prime ideals and hence an isomorphism onto a direct summand by the argument just given. δ vanishes on its image, and the induced map

$$M^0/A \longrightarrow M^1$$

is also injective modulo all prime ideals by assumption, so it is an isomorphism onto a direct summand again, proving ii.

Under the hypothesis of iii, the rank of δ is at least $r_0 - 1$ at every point of S where r^i is the rank of M^i. So the Fitting ideal

$$\operatorname{im}\left(\Lambda^{r^0-1}M^1 \otimes \Lambda^{r^0-1}(M^0)^{\text{dual}} \longrightarrow A\right)$$

is not contained in any maximal ideal, i. e. is all of A. Let Z be the closed subscheme of S defined by the previous Fitting ideal

$$\operatorname{im}\left(\Lambda^{r^0}M^1 \otimes \Lambda^{r^0}(M^0)^{\text{dual}} \longrightarrow A\right).$$

We check that Z has the universal property iii.

Let B be an A-algebra, and denote by $f : T \to S$ the corresponding affine morphism. By [Eis95, Proposition 20.8], f factors through Z if and only if the cokernel of $\delta \otimes_A B$ is locally free of rank $r^1 - r^0 + 1$. But the latter is satisfied if and only if $\mathcal{H}om(f^*E, f^*E')$ is cohomologically flat and $p_{T,*}\mathcal{H}om(f^*E, f^*E')$ is a line bundle on T. □

1.2 Ordered extensions

We keep the assumptions of the previous section: X is a smooth projective curve over $k = \bar{k}$ with a marked point P, and we still fix the length N of the filtrations belonging to quasiparabolic structures.

Remark 1.2.1 In this section and in the next one, it is perfectly possible to take $N = 1$. In this case, a quasiparabolic bundle is nothing but a vector bundle, so everything proved about multiple extensions of quasiparabolic bundles will contain statements about vector bundles as a special case.

Definition 1.2.2 *Let $E^1, \ldots, E^L$ be quasiparabolic bundles over $X \times_k S$ where S is a k-scheme.*

i) An ordered extension *of $E^1, \ldots, E^L$ is a quasiparabolic bundle E over $X \times_k S$ together with a chain of (quasiparabolic) subbundles*

$$0 = F^0E \subseteq F^1E \subseteq \ldots \subseteq F^LE = E$$

whose subquotient $F^lE/F^{l-1}E$ is locally in S isomorphic to E^l for all l.

ii) A rigidification *of the ordered extension* $E = \big(E, (F^l E)_{l \le L}\big)$ *is a sequence of isomorphisms*

$$\eta^l : F^l E / F^{l-1} E \xrightarrow{\sim} E^l \qquad l = 1, \dots, L.$$

iii) An isomorphism of rigidified ordered extensions E *and* E' *of* $E^1, \dots, E^L$ *is an isomorphism of quasiparabolic bundles* $E \to E'$ *respecting the given subbundles and isomorphisms.*

Example 1.2.3 *The so-called* trivial extension

$$E^{\mathrm{triv}} := E^1 \oplus E^2 \oplus \ldots \oplus E^L$$

is a rigidified ordered extension of $E^1, \dots, E^L$ *in an obvious way.*

Example 1.2.4 *Choose an open affine covering* $X = U \cup V$ *of our curve* X *and assume that* S *is affine so that*

$$X \times_k S = (U \times_k S) \cup (V \times_k S)$$

is also an open affine covering. With respect to this covering, we assume given a Čech cochain

$$\gamma \in \mathrm{C}^1\left(\bigoplus_{l_1 < l_2} \mathcal{H}om(E^{l_2}, E^{l_1})\right).$$

Then the quasiparabolic bundle

$$E := E^{\mathrm{triv}}\big|_{U \times_k S} \cup_{\mathrm{id}+\gamma} E^{\mathrm{triv}}\big|_{V \times_k S}$$

is a rigidified ordered extension of $E^1, \dots, E^L$ *in a natural way.*

The pullback f^*E of a rigidified ordered extension of $E^1, \dots, E^L$ along a morphism $f : T \to S$ is again a rigidified ordered extension, namely of $f^*E^1, \dots, f^*E^L$. Thus there is a moduli functor of rigidified ordered extensions, defined on the category of schemes over S. The aim of this section is to prove its representability; as usual, this can only be true if our extensions have no automorphisms. This is why we have introduced rigidifications:

Lemma 1.2.5 *Let* $E^1, \dots, E^L$ *be quasiparabolic bundles over* $X \times_k S$ *for some* k*-scheme* S*. Assume that* $\mathrm{Hom}(E^{l_2}, E^{l_1})$ *vanishes fibrewise for all* $l_1 < l_2 \le L$*. Then every automorphism* ϕ *of a rigidified ordered extension* E *of* $E^1, \dots, E^L$ *is the identity.*

Proof: The restriction of ϕ to $F^{L-1}E$ is an automorphism of a rigidified ordered extension of $E^1, \dots, E^{L-1}$. Using induction on L, we may assume that this restriction is the identity. This means that we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{L-1}E & \longrightarrow & E & \longrightarrow & E^L & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \phi} & & \| & & \\ 0 & \longrightarrow & F^{L-1}E & \longrightarrow & E & \longrightarrow & E^L & \longrightarrow & 0. \end{array}$$

We consider $\phi - \mathrm{id} : E \to E$. This morphism vanishes on the subbundle $F^{L-1}E$ of E, so it induces a morphism $E^L \to E$ on the corresponding quotient bundle. Its image must be contained in $F^{L-1}E$ as the composition $E^L \to E \to E^L$ is also zero. But according to corollary 1.1.9.i, there are no nonzero morphisms from E^L to $F^{L-1}E$ because $\mathrm{Hom}(E^L, F^{L-1}E)$ vanishes fibrewise. Hence $\phi = \mathrm{id}$. $\square$

Note that the notion of an isomorphism of rigidified ordered extensions also makes sense over open subschemes U of $X \times_k S$.

Lemma 1.2.6 *If U is an affine open subscheme of $X \times_k S$, then every rigidified ordered extension E of $E^1, \dots, E^L$ is over U isomorphic to the trivial one E^{triv}.*

Proof: The rigidification gives us morphisms $\eta^l : F^l E \longrightarrow E^l$. Their restrictions to U can be extended to morphisms

$$\phi^l : E\big|_U \longrightarrow E^l\big|_U$$

using proposition 1.1.6. The direct sum

$$\bigoplus_l \phi^l : E\big|_U \longrightarrow E^{\mathrm{triv}}|_U$$

is the required isomorphism of rigidified ordered extensions. □

In the case $L = 2$, the set of isomorphism classes of rigidified ordered extensions of E^1, E^2 is just the usual (Yoneda) Ext^1-group of homological algebra. We have the standard relation to cohomology:

Note 1.2.7 *If S is an affine k-scheme, then there is a canonical bijection between the Zariski cohomology group*

$$\mathrm{H}^1(X \times_k S, \mathcal{H}om(E^2, E^1))$$

and the (set of isomorphism classes of) rigidified ordered extensions of the quasiparabolic bundles E^1, E^2 over $X \times_k S$.

Proof: The cohomology group in question is the set of isomorphism classes of torsors under $\mathcal{G} := \mathcal{H}om(E^2, E^1)$, i.e. of sheaves of sets on $X \times_k S$ on which the sheaf of (abelian) groups $\mathcal{G}$ acts principally.

If E is an extension of E^1, E^2, then we have an exact sequence

$$0 \longrightarrow E^1 \longrightarrow E \xrightarrow{\pi} E^2 \longrightarrow 0.$$

Because the functor $\mathcal{H}om(E^2, \,.\,)$ is exact, the inverse image of the identity section under

$$\pi_* : \mathcal{H}om(E^2, E) \longrightarrow \mathcal{H}om(E^2, E^2)$$

is such a $\mathcal{G}$-torsor. This defines a map from the set of extension classes to the cohomology group in question.

The inverse map can be described as follows: Our sheaf of groups $\mathcal{G}$ acts on $E^1 \oplus E^2$ by the formula

$$\phi \cdot (e^1, e^2) := (e^1 + \phi(e^2), e^2)$$

for local sections e^1, e^2 and ϕ of $\check{E}^1, \check{E}^2$ and $\mathcal{G}$. Hence we can twist it with any $\mathcal{G}$-torsor $\mathcal{T}$, obtaining a quasiparabolic bundle

$$E := (E^1 \oplus E^2) \times_{\mathcal{G}} \mathcal{T}$$

which is an extension of E^1, E^2.

One checks easily that these two maps are inverse to each other. □

(If S is just the spectrum of k, then a different proof of this note by embedding into an abelian category can be found in [Hof99].)

Corollary 1.2.8 *Let E^1 and E^2 be quasiparabolic bundles over $X \times_k S$ for some k-scheme S. Assume that $\mathrm{Hom}(E^2, E^1)$ vanishes fibrewise so that the first higher direct image sheaf $\mathrm{R}^1 p_* \mathcal{H}om(E^2, E^1)$ is a vector bundle over S. Then its total space*

$$\mathrm{Tot}_S \left(\mathrm{R}^1 p_* \mathcal{H}om(E^2, E^1)\right) \longrightarrow S$$

is a fine moduli scheme of rigidified ordered extensions of E^1, E^2.

The following main result of this section partially generalizes the preceding statement to ordered extensions of $L \geq 3$ quasiparabolic bundles.

Theorem 1.2.9 *Let S be a k-scheme, and let $E^1, \ldots, E^L$ be quasiparabolic bundles over $X \times_k S$ such that $\mathrm{Hom}(E^{l_2}, E^{l_1})$ vanishes fibrewise for all $l_1 < l_2 \leq L$.*

i) There is a fine moduli scheme

$$\underline{\mathrm{Ext}}(E^L, \ldots, E^1) \longrightarrow S$$

of rigidified ordered extensions of $E^1, \ldots, E^L$.

ii) If S is affine, then there is a non-canonical isomorphism of S-schemes

$$\underline{\mathrm{Ext}}(E^L, \ldots, E^1) \cong \mathrm{Tot}_S \left(\mathrm{R}^1 p_* \bigoplus_{l_1 < l_2} \mathcal{H}om(E^{l_2}, E^{l_1}) \right). \tag{1.3}$$

Proof: Because rigidified ordered extensions of pullbacks of the given bundles have no automorphisms other than the identity, the moduli functor in question is a Zariski sheaf, and it suffices to prove its representability locally in S. We assume without loss of generality that S is affine, say the spectrum of a k-algebra A.

In contrast to the special case $L = 2$ treated above, we need to make the following choices here to construct the isomorphism (1.3):

We choose a covering $X = U \cup V$ of our curve X by two open affine subschemes and consider Čech cochains with respect to the resulting open affine covering

$$X \times_k S = (U \times_k S) \cup (V \times_k S).$$

For each pair of indices $l_1 < l_2 \leq L$, we choose an A-module of 1-cochains

$$\tilde{\mathrm{H}}^1(l_2, l_1) \subseteq \mathrm{C}^1 \left(\mathcal{H}om(E^{l_2}, E^{l_1})\right)$$

that maps isomorphically onto the cohomology group $\mathrm{H}^1 \left(\mathcal{H}om(E^{l_2}, E^{l_1})\right)$; this is possible as the latter is projective as an A-module by corollary 1.1.9.i.

We will prove that the total space of $\bigoplus_{l_1 < l_2} \tilde{\mathrm{H}}^1(l_2, l_1)$ represents our moduli functor. It suffices to show the following: If E is a rigidified ordered extension of $f^*E^1, \ldots, f^*E^L$ where $f : T \to S$ is the affine morphism corresponding to an A-algebra B, then there is a unique cochain

$$\gamma \in \bigoplus_{l_1 < l_2} \tilde{\mathrm{H}}^1(l_2, l_1) \otimes_A B \subseteq \mathrm{C}^1 \left(\bigoplus_{l_1 < l_2} \mathcal{H}om(f^* E^{l_2}, f^* E^{l_1}) \right)$$

such that as a rigidified ordered extension, E is isomorphic to

$$f^* E^{\mathrm{triv}}\big|_{U \times_k T} \cup_{\mathrm{id}+\gamma} f^* E^{\mathrm{triv}}\big|_{V \times_k T}.$$

Just to simplify the notation, we assume $T = S$, i. e. $B = A$ and $f = \mathrm{id}$. (This means no loss of generality because $\tilde{\mathrm{H}}^1(l_2, l_1) \otimes_A B$ also maps isomorphically to $\mathrm{H}^1\left(\mathcal{H}om(f^* E^{l_2}, f^* E^{l_1})\right)$ by cohomological flatness.)

So E is now a rigidified ordered extension of $E^1, \ldots, E^L$. By lemma 1.2.6, we can trivialize it over $U \times_k S$ and over $V \times_k S$. Hence there is a 1-cochain

$$\upsilon \in \mathrm{C}^1\left(\bigoplus_{l_1 < l_2} \mathcal{H}om(E^{l_2}, E^{l_1})\right)$$

such that E is isomorphic to

$$E^{\mathrm{triv}}\big|_{U \times_k S} \cup_{\mathrm{id}+\upsilon} E^{\mathrm{triv}}\big|_{V \times_k S}.$$

Of course, the trivialisations of E over $U \times_k S$ and over $V \times_k S$ are not unique. We can alter them by automorphism $\mathrm{id} + \phi_U$ and $\mathrm{id} + \phi_V$ of the trivial extension over $U \times_k S$ and over $V \times_k S$; here

$$\phi = (\phi_U, \phi_V) \in \mathrm{C}^0\left(\bigoplus_{l_1 < l_2} \mathcal{H}om(E^{l_2}, E^{l_1})\right)$$

is a Čech cochain. If we do this, then the gluing isomorphism $\mathrm{id} + \upsilon$ gets replaced by $(\mathrm{id} + \phi_U) \circ (\mathrm{id} + \upsilon) \circ (\mathrm{id} + \phi_V)^{-1}$. So the theorem is a consequence of the following computation. □

Lemma 1.2.10 *For each Čech cochain $\upsilon \in \mathrm{C}^1\left(\bigoplus_{l_1<l_2} \mathcal{H}om(E^{l_2}, E^{l_1})\right)$, there are unique cochains*

$$\phi = (\phi_U, \phi_V) \in \mathrm{C}^0\left(\bigoplus_{l_1 < l_2} \mathcal{H}om(E^{l_2}, E^{l_1})\right) \quad \text{and} \quad \gamma \in \bigoplus_{l_1 < l_2} \tilde{\mathrm{H}}^1(l_2, l_1)$$

such that the following equation of automorphisms of E^{triv} over $(U \cap V) \times_k S$ holds:

$$(\mathrm{id} + \gamma) \circ (\mathrm{id} + \phi_V) = (\mathrm{id} + \phi_U) \circ (\mathrm{id} + \upsilon) \tag{1.4}$$

Proof: The quasiparabolic bundle E^{triv} has a natural grading; we denote by

$$\mathcal{E}nd^d(E^{\mathrm{triv}}) := \bigoplus_{l=1}^{L-d} \mathcal{H}om(E^{l+d}, E^l)$$

its sheaf of endomorphisms of degree $-d$ for $d = 1, \ldots, L-1$. The component of our equation (1.4) in degree $-d$ reads

$$\gamma^d - \delta(\phi^d) = \upsilon^d + \sum_{\substack{d', d'' \geq 1 \\ d' + d'' = d}} \left(\phi_U^{d'} \circ \upsilon^{d''} - \gamma^{d''} \circ \phi_V^{d'}\right) \tag{1.5}$$

where δ is the Čech coboundary, defined by $\delta(\phi) = \phi_U - \phi_V$.

If $\gamma^1, \phi^1, \ldots, \gamma^{d-1}, \phi^{d-1}$ are given, then the right hand side of (1.5) is determined, and this equation has a unique solution (γ^d, ϕ^d) because δ is injective and $\tilde{\mathrm{H}}^1$ is mapped isomorphically onto its cokernel.

Component by component, this finally shows that the equation (1.4) has a unique solution, too. □

Question 1.2.11 *If S is not affine, does (1.3) still hold? If not, what can be said about the obstruction?*

Remark 1.2.12 Part ii states in particular that locally in S, the moduli scheme $\underline{\mathrm{Ext}}(E^L, \dots, E^1)$ is simply a relative affine space. Its dimension can be expressed in terms of the multiplicity vector

$$m^{(l)} = (r^{(l)}, \check{d}^{(l)}, m_1^{(l)}, \dots, m_N^{(l)}) \quad \text{of } E^l$$

by means of corollary 1.1.8 (and Riemann-Roch); the result is

$$\dim \underline{\mathrm{Ext}}(E^L, \dots, E^1) = -\sum_{l_1<l_2} \chi\left(\mathcal{H}om(E^{l_2}, E^{l_1})\right) =$$
$$\frac{1}{2}\sum_{l_1<l_2}\left((2g-1)r^{(l_1)} \cdot r^{(l_2)} - \sum_{n=1}^{N} m_n^{(l_1)} \cdot m_n^{(l_2)} + \Delta\left(m^{(l_1)}, m^{(l_2)}\right)\right).$$

1.3 Unordered extensions

As before, we work over a pointed curve $(X/k, P)$ and study quasiparabolic bundles with a fixed length N of the filtration over P. This section will have analogies with the previous one, the main difference being that it deals with *unordered* extensions which will mean that we allow permutations of the given bundles.

To avoid endless repetitions, we fix the following notation for this section:

S is a scheme of finite type over k. For each element i of a finite index set I having L elements, we assume given a simple quasiparabolic bundle E^i over $X \times_k S$ such that $\mathrm{Hom}(E^i, E^j)$ vanishes fibrewise for all $i \neq j$.

An *ordering* σ of I is a bijection $\sigma : \{1, \dots, L\} \xrightarrow{\sim} I$. We say that E is an ordered extension of $E^{\sigma(1)}, \dots, E^{\sigma(L)}$ if there is a chain of subbundles $(F^l E)_{l \leq L}$ that makes E such an ordered extension; in fact it is unique if it exists:

Note 1.3.1 *Assume that the chain of subbundles*

$$0 = F^0 E \subseteq \ldots \subseteq F^L E = E$$

makes the quasiparabolic bundle E an ordered extension of $E^{\sigma(1)}, \dots, E^{\sigma(L)}$ for some ordering σ of I.

i) $\mathcal{H}om(E, E^{\sigma(L)})$ is cohomologically flat, and its direct image sheaf along $p : X \times_k S \to S$ is a line bundle $\mathcal{L}$ on S. The canonical morphism

$$E \longrightarrow E^{\sigma(L)} \otimes_{\mathcal{O}_S} \mathcal{L}^{\mathrm{dual}}$$

is surjective with kernel $F^{L-1}E$.

ii) The subbundles $F^l E$ are uniquely determined by σ and E.

Proof: i) Locally in S, we choose an isomorphism

$$\phi : E/F^{L-1}E \xrightarrow{\sim} E^{\sigma(L)}.$$

By our assumptions on $\mathrm{Hom}(E_s^i, E_s^{\sigma(L)})$, every morphism from E_s to $E_s^{\sigma(L)}$ is a scalar multiple of ϕ_s. Hence we can apply corollary 1.1.9.ii; in particular, ϕ is a local generator of $\mathcal{L}$, and i follows.

ii) According to i, $F^{L-1}E$ is determined by E and $E^{\sigma(L)}$. The claim follows by induction. □

The same quasiparabolic bundle can be an ordered extension for different orderings. For example, the direct sum $E^i \oplus E^j$ is an ordered extension of E^i, E^j as well as an ordered extension of E^j, E^i. This example is typical in the following sense:

Proposition 1.3.2 *Assume that S is just the spectrum of k, and let E be an ordered extension of $E^{\tau(1)}, \ldots, E^{\tau(L)}$ for some ordering τ of I.*

i) Each nonzero morphism ϕ from E to an $E^{\tau(l)}$, $l \leq L$, is surjective. Its kernel is an ordered extension of $E^{\tau(1)}, \ldots, E^{\tau(l-1)}, E^{\tau(l+1)}, \ldots, E^{\tau(L)}$.

ii) If E is also an ordered extension of $E^{\sigma(1)}, \ldots, E^{\sigma(L)}$ for another ordering $\sigma \neq \tau$ of I, then there is a sequence of orderings of I

$$\tau = \sigma_0, \sigma_1, \ldots, \sigma_R = \sigma \qquad R \geq 1$$

such that the following conditions are satisfied for all $r < R$:

- *E is an ordered extension of $E^{\sigma_r(1)}, \ldots, E^{\sigma_r(L)}$.*
- *$\sigma_r(l)$ and $\sigma_{r+1}(l)$ differ only for two consecutive numbers $l = l_r, l_r + 1$.*
- *If $(F^l E)_{l \leq L}$ is the chain of subbundles corresponding to σ_r, then the short exact sequence*

$$0 \longrightarrow \frac{F^{l_r}E}{F^{l_r-1}E} \longrightarrow \frac{F^{l_r+1}E}{F^{l_r-1}E} \longrightarrow \frac{F^{l_r+1}E}{F^{l_r}E} \longrightarrow 0$$

splits.

Proof: i) Denote by $(F^l E)_{l \leq L}$ the chain of subbundles that makes E an ordered extension of $E^{\tau(1)}, \ldots, E^{\tau(L)}$. By our assumptions on the E^i, we have

$$\mathrm{Hom}(F^{l-1}E, E^{\tau(l)}) \ = \ 0 \ = \ \mathrm{Hom}(E/F^l E, E^{\tau(l)}).$$

Hence ϕ induces a nonzero morphism from $F^l E/F^{l-1}E$ to $E^{\tau(l)}$. This is an isomorphism because both bundles are simple and isomorphic. So ϕ is surjective, and it induces a splitting of the short exact sequence

$$0 \longrightarrow F^l E/F^{l-1}E \longrightarrow E/F^{l-1}E \longrightarrow E/F^l E \longrightarrow 0.$$

This means that the kernel of ϕ modulo $F^{l-1}E$ is isomorphic to $E/F^l E$.

ii) We argue by induction on L.

If $\sigma(L)$ equals $\tau(L)$, then the surjections $E \to E^{\sigma(L)}$ and $E \to E^{\tau(L)}$ coming from the two extension structures have the same kernel E' due to note 1.3.1.i. We can apply the induction hypothesis to E', and the proposition follows.

So we may assume that there is an $l_0 < L$ with $\sigma(L) = \tau(l_0)$. For fixed L, we use descending induction on l_0. As E is an ordered extension with order τ, we have a short exact sequence

$$0 \longrightarrow E^{\tau(l_0)} \longrightarrow F^{l_0+1}E/F^{l_0-1}E \longrightarrow E^{\tau(l_0+1)} \longrightarrow 0. \tag{1.6}$$

But E is also an ordered extension of $E^{\sigma(1)}, \ldots, E^{\sigma(L-1)}, E^{\sigma(L)} = E^{\tau(l_0)}$, so there is a nonzero morphism $\phi : E \to E^{\tau(l_0)}$. As we have seen in the proof of i, ϕ induces a nonzero morphism

$$F^{l_0+1}E/F^{l_0-1}E \longrightarrow E^{\tau(l_0)}$$

which splits the sequence (1.6). So there is a complement $E'/F^{l_0-1}E$ of $E^{\tau(l_0)}$ in $F^{l_0+1}E/F^{l_0-1}E$. The chain of subbundles

$$0 = F^0E \subseteq \ldots \subseteq F^{l_0-1}E \subseteq E' \subseteq F^{l_0+1}E \subseteq \ldots \subseteq F^LE = E$$

makes E an ordered extension of

$$E^{\tau(1)}, \ldots, E^{\tau(l_0-1)}, E^{\tau(l_0+1)}, E^{\tau(l_0)} = E^{\sigma(L)}, E^{\tau(l_0+2)}, \ldots, E^{\tau(L)}.$$

An application of the induction hypothesis completes the descending induction. □

Definition 1.3.3 *Assume that S is reduced. A quasiparabolic bundle E over $X \times_k S$ is an* unordered extension *of the E^i, $i \in I$, if for each closed point s of S, there is an ordering σ of I such that the fibre E_s is an ordered extension of*

$$E_s^{\sigma(1)}, \ldots, E_s^{\sigma(L)}.$$

For example, $E^{\mathrm{triv}} := \oplus_{i \in I} E^i$ is an unordered extension, the *trivial* one.

Maybe one should emphasize that the orderings σ do not belong to the datum of an unordered extension. An *isomorphism of unordered extensions* is nothing but an isomorphism of quasiparabolic bundles.

Proposition 1.3.4 *Let E be a quasiparabolic bundle over $X \times_k S$ which is an unordered extension of the E^i, $i \in I$, over each closed point s of S. For each ordering σ of I, there is a unique closed subscheme S_σ of S with the following universal property:*

Any k-morphism of finite type $f : T \to S$ factors through S_σ if and only if f^*E is an ordered extension of $f^*E^{\sigma(1)}, \ldots, f^*E^{\sigma(L)}$.

Proof: Note that S is not assumed to be reduced. We proceed by induction on L.

By our assumptions on $\mathrm{Hom}(E_s^i, E_s^j)$, we have

$$\dim \mathrm{Hom}(E_s, E_s^{\sigma(L)}) \leq 1$$

for each closed point s of S. Applying corollary 1.1.9.iii, we get a largest closed subscheme Z of S over which $\mathcal{H}om(E, E^{\sigma(L)})$ is cohomologically flat and its direct image sheaf is a line bundle $\mathcal{L}$ on Z. By proposition 1.3.2.i, the natural morphism

$$E\big|_Z \longrightarrow E^{\sigma(L)}\big|_Z \otimes_{\mathcal{O}_Z} \mathcal{L}^{\mathrm{dual}}$$

is surjective, and we can apply the induction hypothesis to its kernel. The resulting closed subscheme of Z is the S_σ we are looking for. □

Remark 1.3.5 Over non-reduced base schemes, one could try the following definition: E is an unordered extension of the E^i if it is so over every closed point of S and S is the scheme-theoretic union of its closed subschemes S_σ. (The latter means by definition that the intersection of the corresponding ideal sheaves is zero.)

However, this property is not preserved under pullback because scheme-theoretic union does not commute with pullback in general. (Example: The spectrum S of $k[x,y]/(xy)$ is the union of two lines S_1, S_2 in the plane, but the closed subscheme $T \subset S$ given by the ideal $(x-y) \subset k[x,y]/(xy)$ is not the scheme-theoretic union of $S_1 \times_S T$ and $S_2 \times_S T$.)

That is why we avoid to define unordered extensions over non-reduced bases.

Definition 1.3.6 *Let S be reduced. A* rigidification *of an unordered extension E of the E^i, $i \in I$, consists of a rigidification $(\eta^l_\sigma)_{l \leq L}$ of the ordered extension $E|_{S_\sigma}$ for each ordering σ of I such that the following compatibility condition is satisfied:*

Locally in $X \times_k S$, there is a morphism $\tilde{\eta} : E \to E^{\mathrm{triv}}$ whose restriction to S_σ is an isomorphism of rigidified ordered extensions of $E^{\sigma(1)}, \ldots, E^{\sigma(L)}$ for all σ.

As one might guess, an *isomorphism* ϕ between two rigidified unordered extensions E and E' of the E^i is by definition a morphism $\phi : E \to E'$ of quasiparabolic bundles whose restriction to S_σ is an isomorphism of rigidified ordered extensions of $E^{\sigma(1)}, \ldots, E^{\sigma(L)}$ for all σ.

The name 'rigidification' is justified by the observation that every automorphism of a rigidified unordered extension E of the E^i is the identity.

Proposition 1.3.7 *Assume that S is reduced. Let $\left(\eta^l_\sigma\right)_{l,\sigma}$ and $\left(\omega^l_\sigma\right)_{l,\sigma}$ be two rigidifications of the same unordered extension E. Then there is a unique collection $(f^i)_{i \in I}$ of invertible functions f^i on S such that*

$$\omega^l_\sigma = f^{\sigma(l)}\big|_{S_\sigma} \cdot \eta^l_\sigma$$

holds for all σ and all l.

Proof: For each σ and each l, we have an automorphism

$$\omega^l_\sigma \circ \left(\eta^l_\sigma\right)^{-1} : E^{\sigma(l)}_\sigma \longrightarrow E^{\sigma(l)}_\sigma.$$

What has to be proved is that there is a unique automorphism ϕ^i of E^i for each $i \in I$ such that

$$\phi^{\sigma(l)}\big|_{S_\sigma} = \omega^l_\sigma \circ \left(\eta^l_\sigma\right)^{-1}$$

holds for all σ and all l. As E^i is simple, ϕ^i will automatically be the multiplication with an invertible function f^i on S, and the proposition will follow.

The uniqueness of ϕ^i is obvious, even over open subschemes of $X \times_k S$, so it suffices to show the existence of ϕ^i locally. But by the compatibility condition, we have locally in $X \times_k S$ an automorphism

$$\tilde{\omega} \circ (\tilde{\eta})^{-1} : E^{\mathrm{triv}} \longrightarrow E^{\mathrm{triv}};$$

we can take for $(\phi^i)_{i \in I}$ its image under the canonical morphism

$$\mathcal{E}nd(E^{\mathrm{triv}}) \longrightarrow \bigoplus_{i \in I} \mathcal{E}nd(E^i)$$

that sends each matrix to its entries on the diagonal. □

The pullback of a rigidified unordered extension of the E^i is again a rigidified unordered extension, namely of the pullbacks of the E^i. This defines a moduli functor on the category of reduced schemes of finite type over S which is in fact representable:

Theorem 1.3.8 *Assume given the reduced scheme S of finite type over k and the finite set $\{E^i : i \in I\}$ of simple quasiparabolic bundles over $X \times_k S$ such that $\mathrm{Hom}(E^i, E^j)$ vanishes fibrewise for all $i \neq j$.*

i) Locally in S, every unordered extension E of the E^i has a rigidification.

ii) There is a fine moduli scheme

$$\underline{\mathrm{Ext}}\{E^i : i \in I\} \stackrel{u}{\longrightarrow} S$$

of rigidified unordered extension of the E^i, $i \in I$.

iii) If S is affine, then there is a non-canonical isomorphism of S-schemes

$$\underline{\mathrm{Ext}}\{E^i : i \in I\} \stackrel{\sim}{\longrightarrow} \bigcup_{\sigma} \mathrm{Tot}_S \left(\bigoplus_{l_1 < l_2 \leq L} \mathrm{R}^1 p_* \mathcal{H}om(E^{\sigma(l_2)}, E^{\sigma(l_1)}) \right)$$

where the right hand side is a union of closed subschemes of

$$\mathrm{Tot}_S \left(\bigoplus_{i \neq j} \mathrm{R}^1 p_* \mathcal{H}om(E^i, E^j) \right).$$

Proof: We may assume without loss of generality that S is affine, say the spectrum of a finitely generated k-algebra A. It will be convenient to drop for a while the assumption that S is reduced.

We denote by $\underline{\mathrm{Ext}}\{E^i : i \in I\} \stackrel{u}{\longrightarrow} S$ the S-scheme defined by iii. Note that this is the union of the moduli schemes $\underline{\mathrm{Ext}}(E^{\sigma(L)}, \dots, E^{\sigma(1)})$ of rigidified *ordered* extensions as constructed in the proof of theorem 1.2.9. Furthermore, we can construct a quasiparabolic bundle

$$E^{\mathrm{univ}} \quad \text{over} \quad X \times_k \underline{\mathrm{Ext}}\{E^i : i \in I\}$$

whose restriction to $\underline{\mathrm{Ext}}(E^{\sigma(L)}, \dots, E^{\sigma(1)})$ is a universal rigidified ordered extension for all orderings σ of I; this is done as follows:

For each pair $i \neq j \in I$, we choose an A-module of Čech cochains

$$\tilde{\mathrm{H}}^1(i,j) \subseteq \mathrm{C}^1 \left(\mathcal{H}om(E^i, E^j) \right)$$

that maps isomorphically onto $\mathrm{H}^1(\mathcal{H}om(E^i, E^j))$. (Like in the ordered case, we cover $X \times_k S$ by two open affine subschemes $U \times_k S$ and $V \times_k S$.) We denote by

$$\gamma^{\mathrm{univ}} \in \bigoplus_{i \neq j} u^* \tilde{\mathrm{H}}^1(i,j) \subseteq \mathrm{C}^1 \left(\bigoplus_{i \neq j} \mathcal{H}om(u^* E^i, u^* E^j) \right)$$

the unique cochain whose cohomology class is the tautological section in the pullback of the vector bundle $\bigoplus_{i\neq j} \mathrm{R}^1 p_* \mathcal{H}om(E^i, E^j)$ to the subscheme $\underline{\mathrm{Ext}}\{E^i : i \in I\}$ of its own total space. Now we can define

$$E^{\mathrm{univ}} := u^* E^{\mathrm{triv}}\big|_{U\times_k S} \cup_{\mathrm{id}+\gamma^{\mathrm{univ}}} u^* E^{\mathrm{triv}}\big|_{V\times_k S}.$$

Note that this construction commutes with affine base change due to cohomological flatness. (The main point here is that $\tilde{\mathrm{H}}^1(i,j) \otimes_A B$ also maps isomorphically onto $\mathrm{H}^1\left(\mathcal{H}om(f^* E^i, f^* E^j)\right)$ if $f : T \longrightarrow S$ is an affine morphism corresponding to an A-algebra B.) We shall use this freely to simplify the situation.

A major step towards the proof of the theorem is the following:

Proposition 1.3.9 *Assume given an ordering σ of I and a section*

$$c : S \longrightarrow \underline{\mathrm{Ext}}\{E^i : i \in I\}$$

of u. If $E := c^* E^{\mathrm{univ}}$ is an ordered extension of $E^{\sigma(1)}, \ldots, E^{\sigma(L)}$, then c factors through the closed subscheme

$$\underline{\mathrm{Ext}}(E^{\sigma(L)}, \ldots, E^{\sigma(1)}) \subseteq \underline{\mathrm{Ext}}\{E^i : i \in I\}.$$

Proof: In a first step, let us assume that S is just the spectrum of k. Then c certainly factors through $\underline{\mathrm{Ext}}(E^{\tau(L)}, \ldots, E^{\tau(1)})$ for some ordering τ of I. In particular, E is also an ordered extension with order τ; we denote the resulting chain of subbundles by $(F^l E)_{l \leq L}$. Due to proposition 1.3.2, we may assume without loss of generality that there is an $l_0 < L$ such that

$$\tau(l) = \begin{cases} \sigma(l) & \text{for} \quad l \neq l_0, l_0+1 \\ \sigma(l_0+1) =: j & \text{for} \quad l = l_0 \\ \sigma(l_0) =: i & \text{for} \quad l = l_0+1 \end{cases}$$

holds and the short exact sequence

$$0 \longrightarrow E^j \longrightarrow F^{l_0+1}E/F^{l_0-1}E \longrightarrow E^i \longrightarrow 0$$

splits. By construction of E^{univ}, we have

$$F^{l_0+1}E/F^{l_0-1}E = \left(E^i \oplus E^j\right)\big|_U \cup_{\mathrm{id}+\gamma(i,j)} \left(E^i \oplus E^j\right)\big|_V$$

where $\gamma(i,j)$ is one component of the pullback $\gamma := c^* \gamma^{\mathrm{univ}}$. Now the splitting means that $\gamma(i,j)$ is a Čech coboundary; hence it is zero by the choice of $\tilde{\mathrm{H}}^1(i,j)$. This precisely means that c also factors through $\underline{\mathrm{Ext}}(E^{\sigma(L)}, \ldots, E^{\sigma(1)})$.

The second step is to consider general S. By the special case just treated, the restriction of c to a closed subscheme S' defined by a nilpotent ideal $\mathfrak{n} \subset A$ factors through $\underline{\mathrm{Ext}}(E^{\sigma(L)}, \ldots, E^{\sigma(1)})$. Using induction, we may assume without loss of generality $\mathfrak{n}^2 = 0$.

There is a rigidification of the ordered extension $c^* E^{\mathrm{univ}}$ that extends the rigidification over S' coming from $\underline{\mathrm{Ext}}(E^{\sigma(L)}, \ldots, E^{\sigma(1)})$. According to the proof of theorem 1.2.9, this means that there is a cochain

$$\gamma \in \bigoplus_{l_1 < l_2} \tilde{\mathrm{H}}^1(\sigma(l_2), \sigma(l_1))$$

and an isomorphism of quasiparabolic bundles

$$c^* E^{\mathrm{univ}} \cong E^{\mathrm{triv}}\big|_{U\times_k S} \cup_{\mathrm{id}+\gamma} E^{\mathrm{triv}}\big|_{V\times_k S}$$

which is over S' an isomorphism of rigidified ordered extensions. Hence there are bundle automorphisms $\mathrm{id} + \phi_U$ of $E^{\mathrm{triv}}\big|_{U \times_k S}$ and $\mathrm{id} + \phi_V$ of $E^{\mathrm{triv}}\big|_{V \times_k S}$ such that

$$\mathrm{id} + c^*\gamma^{\mathrm{univ}} = (\mathrm{id} + \phi_U) \circ (\mathrm{id} + \gamma) \circ (\mathrm{id} + \phi_V)^{-1}$$

holds. By the uniqueness part of theorem 1.2.9, ϕ_U and ϕ_V vanish modulo $\mathfrak{n}$; this reduces the equation to

$$c^*\gamma^{\mathrm{univ}} = \gamma + \delta(\phi) + \phi_U \circ \gamma - \gamma \circ \phi_V$$

where we consider $\phi = (\phi_U, \phi_V) \in \mathrm{C}^0\left(\mathcal{E}nd(E^{\mathrm{triv}})\right)$ as a Čech cochain. With respect to the grading

$$\mathcal{E}nd^d(E^{\mathrm{triv}}) := \bigoplus_{l_2 - l_1 = d} \mathcal{H}om\left(E^{\sigma(l_2)}, E^{\sigma(l_1)}\right),$$

the component in degree $d < 0$ of our equation is

$$(c^*\gamma^{\mathrm{univ}})^d = \delta(\phi^d) + \sum_{d'=1-L}^{d-1} \left(\phi_U^{d'} \circ \gamma^{d-d'} - \gamma^{d-d'} \circ \phi_V^{d'}\right). \tag{1.7}$$

We show by induction that $(c^*\gamma^{\mathrm{univ}})^d$ and ϕ^d both vanish for $d < 0$.

Indeed, let us assume that $(c^*\gamma^{\mathrm{univ}})^{1-L}, \phi^{1-L}, \ldots, (c^*\gamma^{\mathrm{univ}})^{d-1}, \phi^{d-1}$ vanish already. Then (1.7) simplifies to

$$(c^*\gamma^{\mathrm{univ}})^d = \delta(\phi^d).$$

The left hand side of this lies in the direct sum of some $\tilde{\mathrm{H}}^1(i,j)$, so it can only be a coboundary if it vanishes. As δ is injective here, $\phi^d = 0$ also follows and the induction is complete.

This vanishing of components of $c^*\gamma^{\mathrm{univ}}$ precisely means that c factors through the closed subscheme $\underline{\mathrm{Ext}}(E^{\sigma(L)}, \ldots, E^{\sigma(1)})$. □

As a consequence of proposition 1.3.9, we can construct classifying morphisms for unordered extensions by gluing the classifying morphisms of ordered extensions as follows:

Corollary 1.3.10 *Assume that S is reduced, and let E be an unordered extension of the E^i, $i \in I$. Locally in S, there is a section*

$$c : S \longrightarrow \underline{\mathrm{Ext}}\{E^i : i \in I\}$$

of u such that c^*E^{univ} is isomorphic to E as a quasiparabolic bundle.

Proof: By proposition 1.3.4, S is the union of finitely many closed subschemes S_n over which E is an ordered extension of $E^{\sigma_n(1)}, \ldots, E^{\sigma_n(L)}$ for some ordering σ_n of I. (Union always means scheme-theoretic union here.) We proceed by induction.

Assume that we already have a morphism of S-schemes

$$c_{n-1} : S_1 \cup \ldots \cup S_{n-1} \longrightarrow \underline{\mathrm{Ext}}\{E^i : i \in I\}$$

and an isomorphism of quasiparabolic bundles

$$\phi_{n-1} : c_{n-1}^* E^{\mathrm{univ}} \xrightarrow{\sim} E\big|_{S_1 \cup \ldots \cup S_{n-1}}.$$

Due to proposition 1.3.9, the restriction of c_{n-1} to the scheme-theoretic intersection

$$S' := (S_1 \cup \ldots \cup S_{n-1}) \times_S S_n$$

factors through the closed subscheme $\underline{\mathrm{Ext}}(E^{\sigma_n(L)}, \ldots, E^{\sigma_n(1)})$. This gives us a rigidification of the ordered extension $E\big|_{S'}$; replacing S by an affine open subscheme, we can extend it to a rigidification of the ordered extension $E\big|_{S_n}$. We get a morphism of S-schemes

$$\tilde{c} : S_n \longrightarrow \underline{\mathrm{Ext}}(E^{\sigma_n(L)}, \ldots, E^{\sigma_n(1)}) \subseteq \underline{\mathrm{Ext}}\{E^i : i \in I\}$$

and an isomorphism of quasiparabolic bundles

$$\tilde{\phi} : \tilde{c}^* E^{\mathrm{univ}} \xrightarrow{\sim} E\big|_{S_n}$$

which coincide with c_{n-1} and ϕ_{n-1} over S'. By the lemma given below, we can glue $\tilde{c}$ to c_{n-1} and $\tilde{\phi}$ to ϕ_{n-1}, obtaining a morphism of S-schemes

$$c_n : S_1 \cup \ldots \cup S_n \longrightarrow \underline{\mathrm{Ext}}\{E^i : i \in I\}$$

and an isomorphism

$$\phi_n : c_n^* E^{\mathrm{univ}} \xrightarrow{\sim} E\big|_{S_1 \cup \ldots \cup S_n}.$$

That completes the induction. □

Lemma 1.3.11 *Assume that a k-scheme T is the union of two closed subschemes T_1 and T_2. Then we can glue morphisms in the following sense:*

i) If $f_1 : T_1 \to T'$ and $f_2 : T_2 \to T'$ are two k-morphisms into another scheme T'/k whose restrictions to the scheme-theoretic intersection $T_1 \times_T T_2$ coincide, then there is a unique k-morphism $f : T \to T'$ that restricts to f_1 and to f_2.

ii) Let E' and E'' be quasiparabolic bundles over $X \times_k T$. If

$$\psi_1 : E\big|_{T_1} \longrightarrow E'\big|_{T_1} \quad \textit{and} \quad \psi_2 : E\big|_{T_2} \longrightarrow E'\big|_{T_2}$$

are two morphisms that coincide when restricted to $T_1 \times_T T_2$, then there is a unique morphism $\psi : E \to E'$ that restricts to ψ_1 and to ψ_2.

Proof: i) We assume without loss of generality that T and T' are affine, say the spectra of rings B and B'. Let $\mathfrak{b}_n \subseteq B$ be the ideal corresponding to T_n. By definition, $T = T_1 \cup T_2$ means $\mathfrak{b}_1 \cap \mathfrak{b}_2 = (0)$.

We consider an arbitrary element $b' \in B'$ and choose elements $b_n \in B$ such that the residue class of b_n modulo $\mathfrak{b}_n$ is $f_n^* b'$. Because f_1 and f_2 coincide on $T_1 \times_T T_2$, the difference $b_1 - b_2$ is of the form $d_1 + d_2$ with $d_n \in \mathfrak{b}_n$. Hence $b := b_1 - d_1 = b_2 + d_2 \in B$ is a common representative of the two residue classes $f_1^* b'$ and $f_2^* b'$. In fact it is the only common representative, so $f^* b' := b$ defines the unique morphism f whose restrictions are f_1 and f_2.

ii) follows from i by considering the two morphisms

$$X \times_k T_n \longrightarrow \mathrm{Tot}_{X \times_k T}\left(\mathcal{H}om(E', E'')\right) \qquad n = 1, 2$$

associated to ψ_1 and ψ_2. □

Now theorem 1.3.8 follows from corollary 1.3.10. In fact, part i of the theorem is an immediate consequence, taking into account that the unordered extension E^{univ} is naturally rigidified.

It remains to check the universality assertion ii of the theorem. By affine base change, it suffices to show that every rigidified unordered extension E of the E^i is isomorphic to the pullback of E^{univ} along a unique section c of u. Now the uniqueness of c follows from the corresponding ordered assertion and proposition 1.3.9. For the existence, 1.3.10 gives us (locally) a section c with $E \cong c^* E^{\mathrm{univ}}$ as quasiparabolic bundles; the two rigidifications differ just by invertible functions f^i according to proposition 1.3.7.

For each $i \in I$, we let a copy of the multiplicative group $\mathbb{G}_m$ act linearly on E^i in the obvious way. By our construction, this induces an action of the torus $\mathbb{G}_m^I$ on $\underline{\mathrm{Ext}}\{E^i : i \in I\}$. If we modify c by an appropriate torus element, we get the required classifying morphism for E. □

Remark 1.3.12 In particular, $\underline{\mathrm{Ext}}\{E^i : i \in I\} \longrightarrow S$ is a Zariski fibration, even in the $\mathbb{G}_m^I$-equivariant sense. More precisely, each point s in S has an open neighborhood $U \subseteq S$ such that there is a $\mathbb{G}_m^I$-equivariant isomorphism of U-schemes

$$\underline{\mathrm{Ext}}\{E^i : i \in I\}\big|_U \xrightarrow{\sim} U \times_k \bigcup_{\sigma} \prod_{l_1 < l_2} F(\sigma(l_2), \sigma(l_1))$$

where $F(i,j)$ is the affine space over k of dimension $-\chi(\mathcal{H}om(E_s^i, E_s^j))$ and the union to the right is a union of linear subspaces of $\prod_{i \neq j} F(i,j)$. Here the torus $\mathbb{G}_m^I/k$ acts linearly on $F(i,j)$ in such a way that $(g^i)_{i \in I}$ acts as multiplication with the scalar g^j/g^i.

Chapter 2

Application to the Boden-Hu conjecture

2.1 Parabolic bundles and their moduli schemes

Like in the previous chapter, we assume given a pointed curve (X, P) over a field $k = \bar{k}$ and consider quasiparabolic bundles whose filtrations at P have a fixed length N. As is well known, there is a notion of stability for such bundles, but it depends on some extra parameters:

Definition 2.1.1 *A* weight vector

$$\alpha = (\alpha_1, \ldots, \alpha_N)$$

is a sequence of real numbers satisfying

$$0 \leq \alpha_1 < \ldots < \alpha_N < 1.$$

Definition 2.1.2 *Let S be a scheme over k. A* parabolic bundle *E over $X \times_k S$ is a quasiparabolic bundle E over $X \times_k S$ together with a weight vector α.*

By definition, the pullback (resp. a subbundle, resp. a quotient bundle) of a parabolic bundle E with weight vector α is the pullback (resp. a subbundle, resp. a quotient bundle) of the underlying quasiparabolic bundle together with the same weight vector α.

Remark 2.1.3 In some texts, e. g. in [BoHu95], it is assumed that all multiplicities are nonzero; then parabolic subbundles and quotients may have a smaller number of weights. However, this text will stick to the point of view that the number of weights is fixed, but multiplicities may be zero. Both views are closely related because there is an obvious way to remove zero multiplicities.

If E and E' are parabolic bundles with the same weight vector, then a morphism $E \longrightarrow E'$ is nothing but a morphism of the underlying quasiparabolic bundles. These are the only morphisms of parabolic bundles that will be considered in this text.

We define the scalar product $m \cdot \alpha$ of a multiplicity vector $m = (r, \check{d}, m_1, \ldots, m_N)$ and a weight vector α by the formula

$$m \cdot \alpha := m_1\alpha_1 + \cdots + m_N\alpha_N.$$

Definition 2.1.4 *The* degree *of a parabolic bundle E with weight vector α is*

$$\deg(E) = \deg_\alpha(E) := \check{d} + m \cdot \alpha$$

where $m = (r, \check{d}, m_1, \ldots, m_N)$ *is the multiplicity vector of* E*. If* E *is nonzero, then the* slope *of* E *is*

$$\mu(E) = \mu_\alpha(E) := \deg_\alpha(E)/r.$$

Definition 2.1.5 *A nonzero parabolic bundle* E *over* X *with weight vector* α *is called* stable *(resp.* semistable*) if*

$$\mu_\alpha(E') < \mu_\alpha(E) \quad (\textit{resp.} \leq)$$

holds for all proper subbundles E' *of* E*.*

Whenever we want to mention α, we refer to these properties as α-stability and α-semistability.

The following Jordan-Hölder type theorem is well known:

Proposition 2.1.6 *Each* α*-semistable parabolic bundle* E *over* X *possesses a chain of subbundles which makes it an ordered extension of* α*-stable parabolic bundles* $E^1, \ldots, E^L$*. The* E^l *are unique up to reordering and isomorphisms.*

Proof: e. g. [Ses82, Troisième Partie, Théorème 12] □

We call the parabolic bundles E^l above the *stable composition factors* of E. Two semistable parabolic bundles over X with the same weight vector α are called *S-equivalent* if their stable composition factors coincide (up to reordering and isomorphisms).

If E is a semistable parabolic bundle over X with weight vector α and $E^i, i \in I$, are its stable composition factors, then the multiplicity vectors of the E^i form an α-partition of the multiplicity vector of E in the following sense:

Definition 2.1.7 *Let* α *be a weight vector. An* α-partition *of a multiplicity vector* $m = (r, \check{d}, m_1, \ldots, m_N)$ *is a collection indexed by some finite set* I

$$\xi = \{m^i : i \in I\}$$

consisting of multiplicity vectors

$$m^i = (r^i, \check{d}^i, m_1^i, \ldots, m_N^i)$$

such that m *is the sum of the* m^i*, each* r^i *is nonzero and the equation*

$$\frac{\check{d} + m \cdot \alpha}{r} = \frac{\check{d}^i + m^i \cdot \alpha}{r^i}$$

holds for all $i \in I$*. The* length $|\xi|$ *of* ξ *is the cardinality of* I*.*

As one might guess, an *ordered* α*-partition* of m is a sequence $(m^1, \ldots, m^L)$ of multiplicity vectors such that $\xi = \{m^1, \ldots, m^L\}$ is an α-partition in the sense just defined. We also call it an *ordering* of ξ.

Definition 2.1.8 *Let* S *be a scheme of finite type over* k*. A parabolic bundle* E *over* $X \times_k S$ *is* stable *(resp.* semistable*) if the fibre* E_s *is stable (resp. semistable) for every closed point* s *on* S*.*

We assume that the genus g of our curve X is at least two and denote by

$$M(m)^\alpha$$

the moduli scheme of semistable parabolic bundles with multiplicity vector m and weight vector α. (For the construction of this scheme and its properties, see [MeSe80], [Ses82] or [Bho89].) It is a normal projective scheme over k endowed with a morphism of functors that assigns to every semistable parabolic bundle

$$E \quad \text{over} \quad X \times_k S$$

with the given weight and multiplicity vector a so-called classifying morphism

$$c = c(E) : S \longrightarrow M(m)^\alpha.$$

This induces a bijection between the k-points of $M(m)^\alpha$ and the S-equivalence classes of semistable parabolic bundles over X with the given weight and multiplicity vector.

One has an open subscheme

$$M(m)^{\alpha-\text{stab}} \subseteq M(m)^\alpha$$

corresponding to stable bundles. It is known to be non-empty (because $g \geq 2$) and smooth of dimension

$$\left(g - \frac{1}{2}\right) r^2 - \frac{1}{2}\left(\sum_{n=1}^{N} m_n^2\right) + 1$$

over k. If our multiplicity vector $m = (r, \check{d}, m_1, \ldots, m_N)$ satisfies

$$\gcd(r, \check{d}, m_1, \ldots, m_N) = 1,$$

then the arguments of [New78, Chapter 4, §5] show the existence of a Poincaré bundle

$$\mathcal{P} = \mathcal{P}(m)^\alpha \quad \text{over} \quad X \times_k M(m)^{\alpha-\text{stab}}.$$

This stable parabolic bundle is characterized by the following property:

For each α-stable parabolic bundle E over $X \times_k S$ with multiplicity vector m, the classifying morphism

$$c = c(E) : S \longrightarrow M(m)^{\alpha-\text{stab}}$$

satisfies

$$E \otimes_{\mathcal{O}_S} \mathcal{L} \cong c^*\mathcal{P}$$

for some line bundle $\mathcal{L}$ on S, and $c(E)$ is the only k-morphism for which this holds.

The moduli scheme $M(m)^\alpha$ can be stratified according to the α-partitions of m that the parameterized semistable parabolic bundles induce. At least over $k = \mathbb{C}$, we can – under some additional assumptions – identify the strata with products of lower-dimensional moduli spaces, cf. [BoHu95, Section 4]. More precisely, one has the following:

Proposition 2.1.9 *Let $g \geq 2$ and $k = \mathbb{C}$. Let a weight vector α and a multiplicity vector m be given such that the multiplicities m_n are all equal to one and such that the resulting parabolic degree $\check{d} + m \cdot \alpha$ is zero. For each α-partition*

$$\xi = \{m^i : i \in I\}$$

of m, the classifying morphism of the direct sum of the Poincaré bundles $\mathcal{P}(m^i)^\alpha$

$$c : \prod_{i \in I} M(m^i)^{\alpha-\text{stab}} \longrightarrow M(m)^\alpha$$

is an isomorphism onto a locally closed subscheme

$$\Sigma_\xi^\alpha \subseteq M(m)^\alpha.$$

As each closed point of $M(m)^\alpha$ lies on precisely one stratum Σ_ξ^α by the Jordan-Hölder theorem, this is in fact a stratification.

2.2 The Boden-Hu desingularisation

We restrict ourselves to weight vectors of a fixed length N lying in the interior of the weight space

$$\overset{\circ}{W} = \overset{\circ}{W}(N, s) := \{\alpha \in \mathbb{R}^N : 0 < \alpha_1 < \ldots < \alpha_N < 1 \quad \text{and} \quad \sum_{n=1}^{N} \alpha_n = s\}.$$

Here s will always be a fixed positive *integer* with $s < N$. Furthermore, we fix the multiplicity vector

$$\underline{\mathbf{1}} := (N, -s, 1, \ldots, 1)$$

so that the parabolic bundles with these multiplicities and weights will have parabolic degree zero, and their rank equals the number of weights N.

Remark 2.2.1 The most important consequence of this special multiplicity vector $\underline{\mathbf{1}}$ is the following: If a semistable parabolic bundle over X has multiplicity vector $\underline{\mathbf{1}}$, then the multiplicity vectors of its stable composition factors are pairwise distinct; as explained in remark 1.1.3, this implies that the composition factors are pairwise nonisomorphic.

Let us summarize some results of [BoHu95] on the behaviour of the moduli scheme $M(\underline{\mathbf{1}})^\alpha$ if the weight vector α is varied.

α is called *generic* if the only α-partition of $\underline{\mathbf{1}}$ is the trivial one $\xi = \{\underline{\mathbf{1}}\}$. This means that α-stability and α-semistability are equivalent for quasiparabolic bundles with multiplicity vector $\underline{\mathbf{1}}$, so we have only one stratum

$$M(\underline{\mathbf{1}})^\alpha = M(\underline{\mathbf{1}})^{\alpha-\text{stab}}$$

which is both smooth and projective.

We say that a generic $\beta \in \overset{\circ}{W}$ is *near* $\alpha \in \overset{\circ}{W}$ if for all 2^N sequences

$$\epsilon_1, \ldots, \epsilon_N \in \{0, 1\},$$

there is no integer strictly between

$$\epsilon_1\alpha_1 + \cdots + \epsilon_N\alpha_N \qquad \text{and} \qquad \epsilon_1\beta_1 + \cdots + \epsilon_N\beta_N.$$

This means that every β-stable quasiparabolic bundle with multiplicity vector $\underline{\mathbf{1}}$ is α-semistable. In particular this holds for the Poincaré bundle $\mathcal{P}(\underline{\mathbf{1}})^\beta$, so it has a classifying morphism

$$\phi_\beta : M(\underline{\mathbf{1}})^\beta \longrightarrow M(\underline{\mathbf{1}})^\alpha$$

which is a resolution of singularities, cf. [BoHu95, Remark 4.2].

Conjecture 2.2.2 (Boden–Hu) *Near every weight vector $\alpha \in \overset{\circ}{W}$, there is a generic weight vector $\beta \in \overset{\circ}{W}$ such that ϕ_β is a small map.*

Recall from [GoMc83, §6.2] that ϕ_β is called *small* (resp. *semismall*) if the locus where its fibres have dimension $\geq d$ has codimension $> 2d$ (resp. $\geq 2d$) in $M(m)^\alpha$ for all positive integers d. This would imply that the intersection homology groups of $M(\underline{\mathbf{1}})^\alpha$ are equal to the ordinary homology groups of $M(\underline{\mathbf{1}})^\beta$; the latter have been computed in [Hol00].

In the next sections, we shall give counterexamples to this conjecture for all ranks $N \geq 9$, and we will prove the conjecture for $N \leq 8$. This will rely on information about the fibres of ϕ_β. In the remainder of this section, we deduce a description of these fibres; the main result is a corrected version of [BoHu95, Theorem 4.5].

Definition 2.2.3 *Assume given $\alpha, \beta \in \overset{\circ}{W}$. An ordered α-partition*

$$(m^1, \dots, m^L) \quad \text{of} \quad \underline{\mathbf{1}} = (N, -s, 1, \dots, 1)$$

*is called β-*stable *if*

$$(m^1 + \dots + m^l) \cdot \beta < (m^1 + \dots + m^l) \cdot \alpha$$

holds for $l = 1, \dots, L-1$.

Lemma 2.2.4 *Let ξ be an α-partition of $\underline{\mathbf{1}}$ with length $|\xi| = L$. If $\beta \in \overset{\circ}{W}$ is generic near α, then precisely $(L-1)!$ of the $L!$ orderings of ξ are β-stable.*

Proof: Choose one ordering $(m^1, \dots, m^L)$ of ξ, and put

$$d(l) := (m^l + \dots + m^L) \cdot \beta - (m^l + \dots + m^L) \cdot \alpha.$$

One checks easily that the cyclicly permuted ordering

$$(m^l, m^{l+1}, \dots, m^L, m^1, \dots, m^{l-1})$$

is β-stable if and only $d(l)$ is strictly smaller than $d(l')$ for all $l' \neq l$. But no two $d(l)$ are equal because β is generic. So there is a unique minimum among them, i.e. precisely one of the L orderings obtained by cyclic permutation is β-stable. □

Theorem 2.2.5 *Let $k = \mathbb{C}$ be the field of complex numbers, and let (X, P) be a pointed smooth projective curve of genus $g \geq 2$ over k. Assume given two weight vectors*

$$\alpha, \beta \in \overset{\circ}{W}(N, s)$$

such that β is generic and near α. We consider the restriction of

$$\phi_\beta : M(\underline{\mathbf{1}})^\beta \longrightarrow M(\underline{\mathbf{1}})^\alpha \qquad \underline{\mathbf{1}} = (N, -s, 1, \dots, 1)$$

to the inverse image[1] of the stratum

$$\Sigma_\xi^\alpha \subseteq M(\underline{\mathbf{1}})^\alpha$$

that corresponds to an α-partition ξ of $\underline{\mathbf{1}}$.

[1] endowed with the reduced subscheme structure

i) The restriction $\phi_\beta : \phi_\beta^{-1}(\Sigma_\xi^\alpha) \longrightarrow \Sigma_\xi^\alpha$ *is a Zariski-locally trivial fibration. Denote its typical fibre by* F_ξ*.*

ii) The fibre F_ξ *is connected and has* $(L-1)!$ *irreducible components if* $L := |\xi|$ *denotes the length of* ξ*. More precisely, there is a canonical bijection that assigns to every* β*-stable ordering of* ξ

$$\big(m^{(1)}, \ldots, m^{(L)}\big)$$

an irreducible component $F_{(m^{(1)},\ldots,m^{(L)})}$ *of* F_ξ*.*

iii) Each component $F_{(m^{(1)},\ldots,m^{(L)})}$ *of the fibre is a smooth projective variety. It is rational of dimension*

$$\frac{1}{2}\sum_{l_1<l_2}\Big((2g-1)r^{(l_1)}\cdot r^{(l_2)} + \Delta\big(m^{(l_1)}, m^{(l_2)}\big)\Big) + 1 - L$$

where $r^{(l)}$ *is the first entry (the 'rank') of the multiplicity vector* $m^{(l)}$*.*

Proof: We work over the base scheme

$$\Sigma_\xi^\alpha = \prod_{i\in I} M(m^i)^{\alpha-\text{stab}}$$

and denote by $\mathcal{P}^{\alpha,i}$ the quasiparabolic bundle over $X \times_k \Sigma_\xi^\alpha$ obtained by pulling back the Poincaré bundle $\mathcal{P}(m^i)^\alpha$ over $X \times_k M(m^i)^{\alpha-\text{stab}}$. We check that these bundles satisfy the hypothesis of theorem 1.3.8.

In fact, they are simple because they are α-stable. For $i \neq j \in I$, the fibres of $\mathcal{P}^{\alpha,i}$ and of $\mathcal{P}^{\alpha,j}$ over each point of Σ_ξ^α are α-stable of the same slope zero, and they are not isomorphic because their multiplicity vectors m^i and m^j are different. It follows that $\mathrm{Hom}(\mathcal{P}^{\alpha,i}, \mathcal{P}^{\alpha,j})$ does indeed vanish fibrewise.

So we can apply theorem 1.3.8 and get a scheme $\underline{\mathrm{Ext}}\{\mathcal{P}^{\alpha,i} : i \in I\}$ over Σ_ξ^α classifying rigidified unordered extensions of the $\mathcal{P}^{\alpha,i}$. We denote by

$$\underline{\mathrm{Ext}}\{\mathcal{P}^{\alpha,i} : i \in I\}^{\beta-\text{stab}}$$

the open subscheme where the universal extension E^{univ} is β-stable. We get a classifying morphism from this open subscheme to $M(\underline{\mathbf{1}})^\beta$. In fact, each point is mapped to the inverse image of Σ_ξ^α; as the $\underline{\mathrm{Ext}}$-scheme is reduced, the classifying morphism factors through this inverse image, leading to a commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Ext}}\{\mathcal{P}^{\alpha,i} : i \in I\}^{\beta-\text{stab}} & \xrightarrow{\;c\;} & \phi_\beta^{-1}(\Sigma_\xi^\alpha) \\ & {\scriptstyle u}\searrow \quad \swarrow{\scriptstyle \phi_\beta} & \\ & \Sigma_\xi^\alpha. & \end{array}$$

The torus $\mathbb{G}_m^I$ acts on $\underline{\mathrm{Ext}}\{\ldots\}^{\beta-\text{stab}}$ by changing the rigidification. The diagonal $\mathbb{G}_m \subseteq \mathbb{G}_m^I$ acts trivially, so we get an action of the quotient torus

$$\mathcal{T} := \mathbb{G}_m^I/\mathbb{G}_m.$$

Lemma 2.2.6 *$c : \underline{\mathrm{Ext}}\{\mathcal{P}^{\alpha,i} : i \in I\}^{\beta-\mathrm{stab}} \longrightarrow \phi_\beta^{-1}(\Sigma_\xi^\alpha)$ is a (Zariski-locally trivial) principal $\mathcal{T}$-bundle.*

Proof: The main reason is that both schemes parameterize the same bundles, but one of them with and the other without a rigidification. More precisely:

The restriction of $\mathcal{P}(\underline{\mathbf{1}})^\beta$ to $\phi_\beta^{-1}(\Sigma_\xi^\alpha)$ is an unordered extension of the quasiparabolic bundles $\phi_\beta^*\mathcal{P}^{\alpha,i}$. By part i of theorem 1.3.8, we can locally rigidify it. Choosing a rigidification over an open subscheme

$$U \subseteq \phi_\beta^{-1}(\Sigma_\xi^\alpha)$$

determines a section of c over U; using the $\mathcal{T}$-action, we get an equivariant morphism

$$\mathcal{T} \times_k U \longrightarrow c^{-1}(U).$$

We want to show that this is an isomorphism. It suffices to check that it is an isomorphism of functors on the category of reduced schemes S of finite type over k.

So assume given a k-morphism

$$f : S \longrightarrow c^{-1}(U).$$

Then f^*E^{univ} and $(f \circ c)^*\mathcal{P}(\underline{\mathbf{1}})^\beta$ are locally isomorphic as quasiparabolic bundles. By proposition 1.3.7, this implies that f can locally be lifted to $\mathcal{T} \times_k U$.

The local isomorphisms between f^*E^{univ} and $(f \circ c)^*\mathcal{P}(\underline{\mathbf{1}})^\beta$ are unique up to invertible functions on S because both bundles are β-stable and hence simple. Using the uniqueness part of proposition 1.3.7, it follows that the local lifts of f to $\mathcal{T} \times_k U$ are locally unique. Hence they can be glued together to a unique lift of f over all of $c^{-1}(U)$.

This shows that c is indeed a trivial principal $\mathcal{T}$-bundle over U. □

The next step is to describe the β-stable locus in $\underline{\mathrm{Ext}}\{\mathcal{P}^{\alpha,i} : i \in I\}$. It suffices to find its k-rational points. These points correspond to unordered extensions E of quasiparabolic bundles E^i, $i \in I$, over X such that each E^i is α-stable of degree zero with multiplicity vector m^i. Now choose an ordering σ of I such that E is an ordered extension of $E^{\sigma(1)}, \ldots, E^\sigma(L)$, and let

$$0 = F^0E \subseteq F^1E \subseteq \ldots \subseteq F^LE = E$$

be the resulting chain of subbundles. If E is β-stable, then we get

$$\deg_\beta(F^lE) < \deg_\alpha(F^lE) = 0$$

for $l = 1, \ldots, L-1$; this precisely means that σ, considered as an ordering of the α-partition $\xi = \{m^i : i \in I\}$ of our multiplicity vector $\underline{\mathbf{1}}$, is β-stable. The converse is also true:

Note 2.2.7 *Consider every ordering σ of I for which E is an ordered extension of $E^{\sigma(1)}, \ldots, E^{\sigma(L)}$ as an ordering of ξ. If they are all β-stable, then E is also β-stable.*

Proof: Assume the contrary, i. e. that there is a quasiparabolic subbundle

$$0 \neq E' \subsetneq E \quad \text{with} \quad \mu_\beta(E') > \mu_\beta(E) = 0.$$

Because β is near α in the sense defined above, this implies

$$\mu_\alpha(E') \geq \mu_\alpha(E),$$

but E is α-semistable, so equality follows. By the Jordan-Hölder theorem 2.1.6, E' is an ordered extension of some of the E^i's, and E/E' is an ordered extension of the remaining E^i's in some ordering. This gives us a bijection σ for which E is an ordered extension of $E^{\sigma(1)}, \dots, E^{\sigma(L)}$ and such that the ordering σ of ξ is not β-stable. □

Corollary 2.2.8

$$\underline{\mathrm{Ext}}\{\mathcal{P}^{\alpha,i} : i \in I\}^{\beta-\mathrm{stab}} = \underline{\mathrm{Ext}}\{\mathcal{P}^{\alpha,i} : i \in I\} \setminus \bigcup_{\sigma} \underline{\mathrm{Ext}}(\mathcal{P}^{\alpha,\sigma(L)}, \dots, \mathcal{P}^{\alpha,\sigma(1)})$$

where the union is taken over all orderings σ of I which are not β-stable as orderings of $\xi = \{m^i : i \in I\}$.

Corollary 2.2.9 *As a scheme with $\mathcal{T}$-action, $\underline{\mathrm{Ext}}\{\mathcal{P}^{\alpha,i} : i \in I\}^{\beta-\mathrm{stab}}$ is a Zariski-locally trivial fibration over Σ^α_ξ. Call its typical fibre $\tilde{F}_\xi$; it can be described as follows:*

For $i \neq j \in I$, let $F(i,j)$ be the affine space of dimension

$$-\chi\left(\mathcal{H}om(\mathcal{P}^{\alpha,i}, \mathcal{P}^{\alpha,j})\right)$$

over k, endowed with the $\mathcal{T}$-action described in remark 1.3.12. To each ordering σ of I, we associate the linear subspace

$$C_\sigma := \prod_{l_1 < l_2} F(\sigma(l_2), \sigma(l_1)) \subseteq \prod_{i \neq j} F(i,j)$$

(which, by the way, is the typical fibre of $\underline{\mathrm{Ext}}\left(\mathcal{P}^{\alpha,\sigma(L)}, \dots, \mathcal{P}^{\alpha,\sigma(L)}\right)$ according to theorem 1.2.9 and the proof of theorem 1.3.8). Then $\tilde{F}_\xi$ is the reduced locally closed $\mathcal{T}$-invariant subscheme

$$\bigcup_{\sigma} C_\sigma \setminus \bigcup_{\sigma \text{ not } \beta\text{-stable}} C_\sigma \subseteq \prod_{i \neq j} F(i,j).$$

We can form the quotient of this typical fibre by $\mathcal{T}$. More precisely, there is a k-scheme F_ξ and a morphism

$$\tilde{F}_\xi \longrightarrow F_\xi$$

which is a Zariski-locally trivial principal $\mathcal{T}$-bundle. To see this, choose a k-rational point on Σ^α_ξ and define F_ξ to be the fibre of ϕ_β over this point. By lemma 2.2.6, $\tilde{F}_\xi$ is indeed a Zariski-locally trivial $\mathcal{T}$-bundle over F_ξ.

Now let U be an open subscheme of Σ^α_ξ such that $c^{-1}(U)$ is isomorphic to $\tilde{F}_\xi \times U$. Then the latter is a Zariski-locally trivial $\mathcal{T}$-bundle over $\phi_\beta^{-1}(U)$ by lemma 2.2.6 again; because such quotients are unique, this implies

$$\phi_\beta^{-1}(U) \cong F_\xi \times U$$

and proves part i of the theorem.

To verify the remaining assertions ii and iii of the theorem, we study the irreducible components of $\tilde{F}_\xi$. The latter is the union of its closed subschemes

$$\tilde{F}_{(m^{\sigma(1)}, \dots, m^{\sigma(L)})} := \tilde{F}_\xi \cap C_\sigma.$$

If σ is not β-stable as an ordering of ξ, then this subscheme is empty. Otherwise, it is not contained in C_τ for $\tau \neq \sigma$ because the dimension of $F(i,j)$ is always positive. (To see the latter, one can use proposition 1.1.6.i to deduce the estimate

$$\deg \mathcal{H}om(E, E') \leq \operatorname{rk}(E) \cdot \deg_\alpha(E') - \operatorname{rk}(E') \cdot \deg_\alpha(E)$$

for arbitrary quasiparabolic bundles E, E'; together with the Riemann-Roch theorem, this leads to

$$-\chi\left(\mathcal{H}om(\mathcal{P}^{\alpha,i}, \mathcal{P}^{\alpha,j})\right) \geq (g-1) r^i \cdot r^j > 0,$$

as required.) In particular, $\tilde{F}_{(m^{\sigma(1)},\ldots,m^{\sigma(L)})}$ is non-empty for β-stable σ; as it is open in the affine space C_σ, it is irreducible.

Hence the irreducible components of $\tilde{F}_\xi$ are precisely those $\tilde{F}_{(m^{\sigma(1)},\ldots,m^{\sigma(L)})}$ for which σ is β-stable; part ii of the theorem follows. (All fibres of ϕ_β are connected anyway by Zariski's main theorem.)

Now let σ be a β-stable ordering of ξ and put $m^{(l)} := m^{\sigma(l)}$. The component $F_{(m^{(1)},\ldots,m^{(L)})}$ of the typical fibre F_ξ is projective because it is closed in the projective scheme $M(\underline{\mathbf{1}})^\beta$; it is smooth of the claimed dimension because $\tilde{F}_{(m^{(1)},\ldots,m^{(L)})}$ is open in the affine space C_σ whose dimension

$$\frac{1}{2} \sum_{l_1 < l_2} \left((2g-1) r^{(l_1)} \cdot r^{(l_2)} + \Delta\big(m^{(l_1)}, m^{(l_2)}\big)\right)$$

has been computed in remark 1.2.12. It remains to check that $F_{(m^{(1)},\ldots,m^{(L)})}$ is rational.

The open $\mathcal{T}$-invariant subscheme

$$\prod_{l=1}^{L-1} \left[F(\sigma(l+1), \sigma(l)) \setminus \{0\}\right] \times \prod_{l_2 - l_1 \geq 2} F(\sigma(l_2), \sigma(l_1)) \subseteq C_\sigma$$

doesn't intersect the other C_τ's, so its quotient by $\mathcal{T}$ is an open subscheme of $F_{(m^{(1)},\ldots,m^{(L)})}$. But this quotient by $\mathcal{T}$ is a Zariski-locally trivial fibration over the product of projective spaces

$$\prod_{l=1}^{L-1} \mathbb{P}F(\sigma(l+1), \sigma(l))$$

with typical fibre the affine space

$$\prod_{l_2 - l_1 \geq 2} F(\sigma(l_2), \sigma(l_1)).$$

Hence it is rational, and $F_{(m^{(1)},\ldots,m^{(L)})}$ is rational, too. □

Remarks 2.2.10 i) If $M(\underline{\mathbf{1}})^\alpha$ consists of only two strata, then theorem 2.2.5 is contained in [BoHu95, Theorem 3.1].

ii) The theorem 2.2.5 just proved contradicts [BoHu95, Theorem 4.5]; the latter states that the fibres of ϕ_β are always irreducible. What's wrong with the argument given by Boden and Hu?

On page 554, line 8, they claim that the number of γ-stable composition factors of a γ-semistable parabolic bundle E cannot exceed the number of its β-stable composition factors by more than one if β covers γ in the sense they define on page 553. Here is a counterexample to that claim:

Let E be a generic ordered extension of three bundles E^1, E^2, E^3 that are γ-stable of degree zero. Let β cover γ in such a way that

$$\deg_\beta(E^1) < \deg_\beta(E^2) = 0 < \deg_\beta(E^3)$$

holds. Then E is β-stable (because E^2 is neither a subbundle nor a quotient of E, just a subquotient), but it has three γ-stable composition factors.

2.3 Partitions of length two and three

We still fix the number N of weights and a positive *integer* $s < N$, the sum of the weights. For weight vectors

$$\alpha, \beta \in \overset{\circ}{W}(N, s)$$

such that β is generic and near α, we consider the Boden-Hu desingularisation

$$\phi_\beta : M(\underline{\mathbf{1}})^\beta \longrightarrow M(\underline{\mathbf{1}})^\alpha$$

of the projective moduli scheme $M(\underline{\mathbf{1}})^\alpha$ of α-semistable quasiparabolic bundles with multiplicity vector

$$\underline{\mathbf{1}} = (N, -s, 1, \dots, 1).$$

For each α-partition ξ of $\underline{\mathbf{1}}$, theorem 2.2.5 describes the typical fibre F_ξ of ϕ_β over the corresponding stratum

$$\Sigma_\xi^\alpha \subset M(\underline{\mathbf{1}})^\alpha;$$

in particular we can compute the 'deviation of ϕ_β from being small'

$$\mathrm{dev}_\beta(\xi) := 2 \dim F_\xi - \mathrm{codim}\, \Sigma_\xi^\alpha.$$

Using the dimension formula in theorem 2.2.5, the result is

$$\mathrm{dev}_\beta(\xi) = 1 - L + \max \sum_{l_1 < l_2 \leq L} \Delta\big(m^{(l_1)}, m^{(l_2)}\big)$$

where the maximum is taken over all β-stable orderings $(m^1, \dots, m^L)$ of ξ.

We say that β is *small over* ξ if $\mathrm{dev}_\beta(\xi)$ is negative. Clearly ϕ_β is a small map if and only if β is small over each α-partition ξ of the multiplicity vector $\underline{\mathbf{1}}$.

Note that the latter property involves only the weights α and β, but not the curve X any more. (Even its genus g has canceled out in the computation of $\mathrm{dev}_\beta(\xi)$). This section deals with smallness over partitions ξ of length two and three.

Proposition 2.3.1 *Near each weight vector $\alpha \in \overset{\circ}{W}(N, s)$, there is a generic weight vector $\beta \in \overset{\circ}{W}(N, s)$ which is small over all length two α-partitions ξ of the multiplicity vector $\underline{\mathbf{1}} = (N, -s, 1, \dots, 1)$.*

Proof: We choose a real number $\varepsilon > 0$ and define $\gamma \in \mathbb{R}^N$ by

$$\gamma_n := \alpha_n + \varepsilon \left(2N\alpha_n - 2s + N - 2n + 1\right) \qquad \text{for} \quad n = 1, \ldots, N.$$

Consider a length two α-partition ξ of $\underline{\mathbf{1}}$. In other words, we have $\xi = \{m, \underline{\mathbf{1}} - m\}$ for a multiplicity vector

$$m = (r, \check{d}, m_1, \ldots, m_N)$$

for which $m \cdot \alpha = m_1\alpha_1 + \cdots + m_N\alpha_N$ is equal to $-\check{d}$. Recall that r is the sum of the m_n. Using all this, we can compute the value of our alternating bilinear form Δ directly from its definition 1.1.7; we get

$$\begin{aligned}\Delta(m, \underline{\mathbf{1}} - m) &= \Delta(m, \underline{\mathbf{1}}) \\ &= 2r \cdot (-s) + \sum_{n=1}^{N} m_n \cdot (N - n) - 2N \cdot \check{d} - \sum_{n=1}^{N} m_n \cdot (n - 1) \\ &= \sum_n m_n \cdot (2N\alpha_n - 2s + N - 2n + 1)\,.\end{aligned}$$

This means that

$$m \cdot \gamma - m \cdot \alpha = \varepsilon\Delta(m, \underline{\mathbf{1}} - m)$$

holds for all these m.

In particular, this computation works for $m = \underline{\mathbf{1}}$ and proves $\gamma_1 + \ldots + \gamma_N = s$. Hence

$$\gamma \in \overset{\circ}{W}(N, s)$$

if ε was chosen sufficiently small. Let β be a generic weight vector near γ. Then β is also near α if ε is small enough. We check that β is small over all length two α-partitions $\xi = \{m, \underline{\mathbf{1}} - m\}$ of $\underline{\mathbf{1}}$.

If $\Delta(m, \underline{\mathbf{1}} - m)$ vanishes, then *every* generic weight vector near α is small over ξ. If not, we can assume without loss of generality

$$\Delta(m, \underline{\mathbf{1}} - m) < 0$$

(replacing m by $\underline{\mathbf{1}} - m$ if necessary). By the computation above, this implies

$$m \cdot \gamma < m \cdot \alpha.$$

As β is near γ, the same inequality holds for β. So the unique β-stable ordering of ξ is $(m, \underline{\mathbf{1}} - m)$, and β is indeed small over ξ. □

Now let ξ be a length three α-partition of $\underline{\mathbf{1}}$. Choose an ordering (m, m', m'') of ξ and denote by

$$\Delta_1 \leq \Delta_2 \leq \Delta_3$$

the three integers

$$\Delta(m, m'), \quad \Delta(m', m''), \quad \Delta(m'', m),$$

ordered by their magnitude. Δ_1, Δ_2 and Δ_3 do depend mildly on the chosen ordering of ξ: If we permute m, m', m'' cyclicly, they remain unchanged. But if we apply one of the three odd permutations to m, m', m'', then Δ_ν gets replaced by $-\Delta_{4-\nu}$. In particular, the expression

$$\mathrm{I}(\xi) := \Delta_1 - \Delta_3 + |\Delta_2|$$

does not depend on the ordering of ξ, so it is an invariant of ξ.

Proposition 2.3.2 *Let ξ be a length three α-partition of $\underline{\mathbf{1}}$. Then*

$$\operatorname{dev}_\beta(\xi) \geq \mathrm{I}(\xi) - 2$$

holds for all generic β near α. If β is small over the three length two α-partitions $(m, \underline{\mathbf{1}} - m)$ with $m \in \xi$, then we have equality:

$$\operatorname{dev}_\beta(\xi) = \mathrm{I}(\xi) - 2.$$

Proof: If (m, m', m'') is an ordering of ξ, then the three quantities

$$m \cdot \beta - m \cdot \alpha, \quad m' \cdot \beta - m' \cdot \alpha, \quad m'' \cdot \beta - m'' \cdot \alpha$$

are nonzero (as β is generic) and have sum zero (since $\underline{\mathbf{1}} \cdot \beta = \underline{\mathbf{1}} \cdot \alpha$). Choosing the ordering appropriately, we may assume without loss of generality

$$m \cdot \beta < m \cdot \alpha \quad \text{and} \quad m'' \cdot \beta > m'' \cdot \alpha.$$

We distinguish two cases.

If $m' \cdot \beta < m' \cdot \alpha$, then the two β-stable orderings of ξ are

$$(m, m', m'') \quad \text{and} \quad (m', m, m''),$$

and the corresponding deviation is

$$\operatorname{dev}_\beta(\xi) = \Delta(m', m'') - \Delta(m'', m) + |\Delta(m, m')| - 2.$$

Now this expression will not increase if we permute $\Delta(m, m')$, $\Delta(m', m'')$ and $\Delta(m'', m)$ in such a way that the largest one gets the negative sign and the smallest one gets the positive sign. Hence it is at least $\mathrm{I}(\xi) - 2$.

For the equality assertion, observe that the unique β-stable orderings of the three length two α-partitions induced by ξ are

$$(m, m' + m''), \quad (m', m + m''), \quad (m + m', m'').$$

If β is small over these three, the corresponding deviations are negative; this precisely means

$$\Delta(m', m'') \leq \Delta(m, m') \leq \Delta(m'', m).$$

This identifies the Δ_ν and proves $\operatorname{dev}_\beta(\xi) = \mathrm{I}(\xi) - 2$.

The remaining second case $m' \cdot \beta > m' \cdot \alpha$ is treated analogously. We get

$$\operatorname{dev}_\beta(\xi) = \Delta(m, m') - \Delta(m'', m) + |\Delta(m', m'')| - 2 \geq \mathrm{I}(\xi) - 2.$$

Stability of β over the three length two α-partitions is equivalent to

$$\Delta(m, m') \leq \Delta(m', m'') \leq \Delta(m'', m),$$

so we get the required equality in this case. □

2.4 Counterexamples for rank nine and beyond

Consider the rank $N \geq 9$, and choose a positive integer $t \leq N/9$. We construct a weight vector

$$\alpha \in \overset{\circ}{W} (N, 3t+1)$$

as follows: Choose a positive real number $\varepsilon < 1/6$, and let the first $N - 6t - 1$ weights

$$0 < \alpha_1 < \ldots < \alpha_{N-6t-1} < 1/6$$

have sum ε. Assume that the next $3t$ weights satisfy

$$1/6 < \alpha_{N-6t} < \ldots < \alpha_{N-3t-1} < 1/2$$

and have average $1/3$, i. e. sum t. Similarly, let the next $3t$ weights satisfy

$$1/2 < \alpha_{N-3t} < \ldots < \alpha_{N-1} < 5/6$$

and have average $2/3$, i. e. sum $2t$. Finally, put

$$\alpha_N := 1 - \varepsilon.$$

Proposition 2.4.1 *Near the weight vector α just constructed, there is no generic weight vector β such that the Boden-Hu desingularisation*

$$\phi_\beta : M(\underline{\mathbf{1}})^\beta \longrightarrow M(\underline{\mathbf{1}})^\alpha$$

is a semismall map.

Proof: By choice of the weights, the three multiplicity vectors

$$\begin{array}{rcl} m &=& (\; N-6t \;,\; -1 \;,\; 1,\ldots,1 \;,\; 0,\ldots,0 \;,\; 0,\ldots,0 \;,\; 1 \;) \\ m' &=& (\; 3t \;,\; -t \;,\; 0,\ldots,0 \;,\; 1,\ldots,1 \;,\; 0,\ldots,0 \;,\; 0 \;) \\ m'' &=& (\; 3t \;,\; -2t \;,\; \underbrace{0,\ldots,0}_{N-6t-1} \;,\; \underbrace{0,\ldots,0}_{3t} \;,\; \underbrace{1,\ldots,1}_{3t} \;,\; 0 \;) \end{array}$$

form an α-partition $\xi = \{m, m', m''\}$ of $\underline{\mathbf{1}}$. Directly from the definition 1.1.7 of Δ, we get

$$\Delta(m', m'') = 3t^2 \quad \text{and} \quad \Delta(m, m') = \Delta(m'', m) = t(N - 6t) \geq 3t^2,$$

so proposition 2.3.2 gives us

$$\mathrm{dev}_\beta(\xi) \geq \mathrm{I}(\xi) - 2 = 3t^2 - 2$$

for each generic β near α. This is positive, so ϕ_β cannot be semismall. □

2.5 Proof of the conjecture for ranks up to eight

Considering small ranks, it is useful to observe the following:

Note 2.5.1 *Assume given a weight vector* $\alpha \in \overset{\circ}{W}(N, s)$ *and two distinct multiplicity vectors*

$$m = (r, \check{d}, m_1, \ldots, m_N) \quad \textit{and} \quad m' = (r', \check{d}', \ldots, m'_N)$$

that occur in the same α*-partition* ξ *of the multiplicity vector* $\underline{\mathbf{1}} = (N, -s, 1, \ldots, 1)$*.*

i) $\Delta(m, m')$ *is congruent to* $r \cdot r'$ *modulo* 2*.*

ii) If $r = r' = 2$*, then* $\Delta(m, m') = 0$*.*

Proof: i) From its very definition 1.1.7, we get

$$\Delta(m, m') \equiv \sum_{n \neq n'} m_n m'_{n'} = r \cdot r' - \sum_n m_n m'_n$$

modulo two. But $m_n m'_n$ is always zero as we have $m_n + m'_n \leq 1$ because m and m' occur in the same partition of $\underline{\mathbf{1}}$.

ii) Our assumptions on m imply that it has the form

$$m = (2, -1, 0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$$

with $m_n = 1$ for two indices $n = n_1, n_2$, say $n_1 < n_2$, such that

$$\alpha_{n_1} + \alpha_{n_2} = 1$$

holds. The same applies to m'; if we denote by $n'_1 < n'_2$ the two values of n which satisfy $m'_n > 0$, then we also have

$$\alpha_{n'_1} + \alpha_{n'_2} = 1.$$

As the α_n are numbered by their size, this implies

$$n_1 < n'_1 < n'_2 < n_2 \qquad \text{or} \qquad n'_1 < n_1 < n_2 < n'_2;$$

in both cases, $\Delta(m, m') = 2r\check{d}' - 2r'\check{d} = 0$ follows. □

The main result of this section is the following:

Theorem 2.5.2 *If* $N \leq 8$ *and* $\alpha \in \overset{\circ}{W}(N, s)$*, then there is a generic weight vector* β *near* α *which is small over all* α*-partitions* ξ *of the weight vector* $\underline{\mathbf{1}} = (N, -s, 1, \ldots, 1)$*. Consequently, the Boden-Hu conjecture holds for ranks* $N \leq 8$*.*

Proof: By proposition 2.3.1, we can find a generic β near α which is small over all ξ of length two. We claim that such a weight vector β is automatically small over *all* ξ.

If a multiplicity vector $m = (r, \ldots)$ occurs in an α-partition ξ of $\underline{\mathbf{1}}$, then its rank r is at least two (since no weight is an integer). Hence there are no partitions ξ of length greater than four.

If $|\xi| = 4$, then all its four multiplicity vectors must have rank two, so we can apply note 2.5.1.ii and get $\text{dev}_\beta(\xi) = -3$ for all generic β near α, i. e. all possible β are small over ξ.

It remains to treat the case $|\xi| = 3$. Using proposition 2.3.2, the following estimate implies that β is small over these ξ, too. □

Proposition 2.5.3 *If $N \leq 8$ and $\alpha \in \overset{\circ}{W}(N,s)$, then $\mathrm{I}(\xi) < 2$ for all length three α-partitions $\xi = \{m, m', m''\}$ of $\underline{\mathbf{1}}$.*

Proof: Let $\Delta_1 \leq \Delta_2 \leq \Delta_3$ be the three integers $\Delta(m,m')$, $\Delta(m',m'')$ and $\Delta(m'',m)$, ordered by their size. Recall that $\mathrm{I}(\xi) = \Delta_1 - \Delta_3 + |\Delta_2|$ by definition. Applying an odd permutation to m, m', m'' if necessary, we may assume without loss of generality $\Delta_2 \geq 0$; this implies

$$\mathrm{I}(\xi) = \Delta_1 - \Delta_3 + \Delta_2 \leq \Delta_1 \leq \Delta_2 \leq \Delta_3.$$

We consider the rank entries r, r' and r'' of our three multiplicity vectors, defined by $m = (r, \ldots)$, $m' = (r', \ldots)$ and $m'' = (r'', \ldots)$. If at least two of them are equal to two, we can apply note 2.5.1.ii and get

$$\mathrm{I}(\xi) \leq 0.$$

The only remaining case is that the ranks r, r' and r'' are 2, 3 and 3. Permuting m, m' and m'' cyclicly if necessary, we may assume without loss of generality $r = 2$ and $r' = r'' = 3$.

By note 2.5.1.i, $\mathrm{I}(\xi)$ is odd, so it suffices to prove $\mathrm{I}(\xi) < 3$. Let us assume the contrary. Then we have

$$\Delta_1, \Delta_2, \Delta_3 \geq 3.$$

But again by note 2.5.1.i, two of these three integers are even, hence their sum is at least eleven. This contradicts the following lemma, so the proposition follows. □

Lemma 2.5.4 *Assume that the multiplicity vectors*

$$\begin{array}{rcl} m & = & (2, \ -1, \ m_1 \ , \ldots, \ m_8) \\ m' & = & (3, \ \check{d}', \ m'_1 \ , \ldots, \ m'_8) \\ m'' & = & (3, \ \check{d}'', \ m''_1 \ , \ldots, \ m''_8) \end{array}$$

form an α-partition $\xi = \{m, m', m''\}$ of $\underline{\mathbf{1}} = (8, -s, 1, \ldots, 1)$. Then

$$\Delta(m,m') + \Delta(m',m'') + \Delta(m'',m) < 11.$$

Proof: In the definition 1.1.7 of Δ, there are summands involving the rank and degree components of the two multiplicity vectors and summands depending on the multiplicities themselves. Let us denote by Δ_{deg} the sum of the former and by Δ_{mul} the sum of the latter. Then

$$\Delta = \Delta_{\mathrm{deg}} + \Delta_{\mathrm{mul}}$$

as bilinear forms.

With the rank and degree entries given, we can compute Δ_{deg}; the result is

$$\Delta_{\mathrm{deg}}(m,m') + \Delta_{\mathrm{deg}}(m',m'') + \Delta_{\mathrm{deg}}(m'',m) = 2\check{d}'' - 2\check{d}'.$$

Now the sum of $\check{d}'$ and three weights is zero, and the same applies to $\check{d}''$. Hence both are equal to -1 or -2, and we get

$$\Delta_{\mathrm{deg}}(m,m') + \Delta_{\mathrm{deg}}(m',m'') + \Delta_{\mathrm{deg}}(m'',m) \leq 2 \tag{2.1}$$

with equality only for $\check{d}' = -2$ and $\check{d}'' = -1$.

The remaining summands are

$$\begin{aligned}\Delta_{\mathrm{mul}}(m,m') + \Delta_{\mathrm{mul}}(m',m'') + \Delta_{\mathrm{mul}}(m'',m) \;&=\; \sum_{n<n'} m_n m'_{n'} - \sum_{n'<n} m_n m'_{n'} + \\ + \sum_{n'<n''} m'_{n'} m''_{n''} - \sum_{n''<n'} m'_{n'} m''_{n''} \;&+\; \sum_{n''<n} m''_{n''} m_n - \sum_{n<n''} m''_{n''} m_n.\end{aligned}$$

Here we have precisely $2 \cdot 3 + 3 \cdot 3 + 3 \cdot 2 = 21$ nonzero summands; they are all equal to ± 1. We count the summands equal to -1.

There are $2 \cdot 3 \cdot 3 = 18$ triples of indices (n, n', n'') with $m_n = m'_{n'} = m''_{n''} = 1$. As $n < n' < n'' < n$ is impossible, at least one of the three summands

$$m_n m'_{n'}, \quad m'_{n'} m''_{n''} \quad \text{and} \quad m''_{n''} m_n \tag{2.2}$$

occurs with a negative sign. Each negative summand determines a pair of indices which can be extended to one of our 18 triples in at most three ways. Hence the number of negative summands is at least $18/3 = 6$, the number of positive summands is at most $21 - 6 = 15$, and we get

$$\Delta_{\mathrm{mul}}(m,m') + \Delta_{\mathrm{mul}}(m',m'') + \Delta_{\mathrm{mul}}(m'',m) \leq 15 - 6 = 9.$$

If this is a strict inequality, then we are done. So let us assume that we have equality here. Then only one of the three summands in (2.2) can have a negative sign, and it must determine a pair of indices which can be extended to one of our 18 triples in exactly three ways. Hence $m'_{n'} m''_{n''}$ can never have a negative sign, i. e. $n' < n''$ for all indices n', n'' with $m'_{n'} = m''_{n''} = 1$. But $n' < n''$ implies $\alpha_{n'} < \alpha_{n''}$, so we get

$$-\check{d}' = m' \cdot \alpha < m'' \cdot \alpha = -\check{d}''.$$

Hence (2.1) is a strict inequality in this case, and the lemma follows. □

Chapter 3

On Arakelov bundles over arithmetic curves

This chapter is completely independent of the previous ones. We introduce new notation, abandoning the special conventions of the previous sections.

3.1 Notation

Let K be a number field of degree d over $\mathbb{Q}$ and with ring of integers $\mathcal{O}_K$. Let $X = \mathrm{Spec}(\mathcal{O}_K) \cup X_\infty$ be the set of places of K; this might be called an 'arithmetic curve' in the sense of Arakelov geometry. X_∞ consists of r_1 real and r_2 complex places with $r_1 + 2r_2 = d$.

For every place $v \in X$, we endow the corresponding completion K_v of K with the map $|\,.\,|_v : K_v \to \mathbb{R}^{\geq 0}$ defined by $\mu(a \cdot S) = |a|_v \cdot \mu(S)$ for a Haar measure μ on K_v. This is the normalized valuation if v is finite, the usual absolute value if v is real and its square if v is complex. The well known product formula $\prod_{v \in X} |a|_v = 1$ holds for every $0 \neq a \in K$.

Let $\mathcal{O}_v$ be the set of those $a \in K_v$ which satisfy $|a|_v \leq 1$; this is the ring of integers in K_v for finite v and the unit disc for infinite v. $\mathcal{O}_\mathbb{A} := \prod_{v \in X} \mathcal{O}_v$ is a compact neighborhood of 0 in the adele ring $\mathbb{A}$.

We fix a canonical Haar measure λ_v on K_v as follows:

- If v is finite, we normalize by $\lambda_v(\mathcal{O}_v) = 1$.
- If v is real, we take for λ_v the usual Lebesgue measure on $\mathbb{R}$.
- If v is complex, we let λ_v come from the real volume form $dz \wedge d\bar{z}$ on $\mathbb{C}$. In other words, we take twice the usual Lebesgue measure.

This gives us a canonical Haar measure $\lambda := \prod_{v \in X} \lambda_v$ on $\mathbb{A}$. We have $\lambda(\mathbb{A}/K) = \Delta^{1/2}$ where $\Delta = \Delta(K)$ denotes (the absolute value of) the discriminant. More details on this measure can be found in section 2.1 of [Wei82].

3.2 A mean value formula

Proposition 3.2.1 *Let $1 \leq l < n$, and let Φ be an integrable function on the space* $\mathrm{Mat}_{n\times l}(\mathbb{A})$ *of $n \times l$ adele matrices. Then*

$$\int\limits_{\mathrm{Sl}_n(\mathbb{A})/\mathrm{Sl}_n(K)} d\tau(A) \sum_{\substack{M\in \mathrm{Mat}_{n\times l}(K)\\ \mathrm{rk}(M)=l}} \Phi(A\cdot M) = \Delta^{-nl/2} \int\limits_{\mathrm{Mat}_{n\times l}(\mathbb{A})} \Phi\, d\lambda^{n\times l}$$

where the measure τ is induced by the Tamagawa measure on $\mathrm{Sl}_n(\mathbb{A})$ *along the local homeomorphism* $\mathrm{Sl}_n(\mathbb{A}) \to \mathrm{Sl}_n(\mathbb{A})/\mathrm{Sl}_n(K)$.

Proof: With real numbers and integers instead of adeles and elements of K, a similar result has already been stated by Siegel in [Sie45], and an elementary proof is given in [MaRo58]. In the adelic context, the case $l = 1$ is done in section 3.4 of [Wei82], and the general case can be deduced along the same lines from earlier sections of this book. For the convenience of the reader, we recall the main arguments.

First of all, note that the sum on the left hand side is well-defined: If $A \in \mathrm{Sl}_n(\mathbb{A})$ is replaced by another representative of its class modulo $\mathrm{Sl}_n(K)$, then the summands are just permuted. The proposition implicitly claims that this series is absolutely convergent outside a set of measure zero in $\mathrm{Sl}_n(\mathbb{A})/\mathrm{Sl}_n(K)$ and that the sum is integrable. For all this, we may assume without loss of generality that Φ is nonnegative.

Let G be the algebraic group Sl_n over the ground field K, and denote by τ_G the Tamagawa measure on $G(\mathbb{A})$ or any quotient by a discrete subgroup. G acts on the affine space $\mathrm{Mat}_{n\times l}$ by left multiplication, and we denote by

$$H = \left(\begin{array}{ccc|c} 1 & & 0 & \\ & \ddots & & * \\ 0 & & 1 & \\ \hline & 0 & & \mathrm{Sl}_{n-l} \end{array}\right) \subseteq G$$

the stabilizer of the rational point

$$E = \left(\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \hline & 0 & \end{array}\right) \in \mathrm{Mat}_{n\times l}(K).$$

With this notation, the left hand side of the claim becomes

$$\int\limits_{G(\mathbb{A})/G(K)} d\tau_G(A) \sum_{B\in G(K)/H(K)} \Phi(AB\cdot E) = \int\limits_{G(\mathbb{A})/H(K)} \Phi(A\cdot E)\, d\tau_G(A).$$

Observe that H is the semi-direct product of Sl_{n-l} and $\mathrm{Mat}_{l\times(n-l)}$. It follows that $H(\mathbb{A})$ is a unimodular group having a canonical Haar measure, the Tamagawa measure τ_H, because the product of the local measures defined by an algebraic volume form converges by Theorem 2.4.4 of [Wei82]. This theorem also gives us the Tamagawa number of H:

$$\int\limits_{H(\mathbb{A})/H(K)} d\tau_H = 1.$$

Now consider the quotient of G by H. Denote by

$$\mathrm{Mat}_{n\times l}^{\mathrm{rk}=l} \subseteq \mathrm{Mat}_{n\times l}$$

the open subvariety whose K'-valued points for extension fields $K' \supseteq K$ are just the matrices of rank l. Then

$$G/H = \mathrm{Mat}_{n\times l}^{\mathrm{rk}=l}$$

and, more precisely, $G \xrightarrow{\cdot E} \mathrm{Mat}_{n\times l}^{\mathrm{rk}=l}$ is a principal bundle with structure group H: one can cover $\mathrm{Mat}_{n\times l}^{\mathrm{rk}=l}$ by $\binom{n}{l}$ open subvarieties isomorphic to $\mathrm{Gl}_l \times \mathrm{Mat}_{(n-l)\times l}$ and trivialize the bundle explicitly over each of these. Hence we can apply Theorems 2.4.2 and 2.4.3 of [Wei82] and obtain the following:

- $\mathrm{Mat}_{n\times l}^{\mathrm{rk}=l}(\mathbb{A}) = G(\mathbb{A})/H(\mathbb{A})$
- We have a canonical invariant measure, the Tamagawa measure $\tau_{G/H}$, on $\mathrm{Mat}_{n\times l}^{\mathrm{rk}=l}(\mathbb{A})$ (because the product of the local measures defined by an algebraic volume form on $\mathrm{Mat}_{n\times l}^{\mathrm{rk}=l}$ converges).
- $\tau_G = \tau_{G/H} \cdot \tau_H$ which means by definition that

$$\int\limits_{G(\mathbb{A})} \Psi \, d\tau_G = \int\limits_{\mathrm{Mat}_{n\times l}^{\mathrm{rk}=l}(\mathbb{A})} d\tau_{G/H}(A\cdot E) \int\limits_{H(\mathbb{A})} \Psi(AB)\, d\tau_H(B)$$

  holds for every nonnegative measurable function Ψ on $G(\mathbb{A})$. (The right hand side is to be interpreted as the integral of the unique function on $\mathrm{Mat}_{n\times l}^{\mathrm{rk}=l}(\mathbb{A})$ whose value at $A \cdot E$ is equal to $\int_{H(\mathbb{A})} \Psi(AB)\, d\tau_H(B)$ for every $A \in G(\mathbb{A})$.)

In the last equation, Lemma 2.4.1 of [Wei82] enables us to divide out the discrete subgroup $H(K)$. Hence

$$\int\limits_{G(\mathbb{A})/H(K)} \Psi \, d\tau_G = \int\limits_{\mathrm{Mat}_{n\times l}^{\mathrm{rk}=l}(\mathbb{A})} d\tau_{G/H}(A\cdot E) \int\limits_{H(\mathbb{A})/H(K)} \Psi(AB)\, d\tau_H(B)$$

if Ψ is $H(K)$-invariant, and in particular

$$\int\limits_{G(\mathbb{A})/H(K)} \Phi(A\cdot E)\, d\tau_G(A) = \int\limits_{\mathrm{Mat}_{n\times l}^{\mathrm{rk}=l}(\mathbb{A})} \Phi \, d\tau_{G/H}.$$

But by Lemma 3.4.1 of [Wei82], the complement of $\mathrm{Mat}_{n\times l}^{\mathrm{rk}=l}(\mathbb{A})$ in $\mathrm{Mat}_{n\times l}(\mathbb{A})$ has measure zero, and $\tau_{G/H}$ is just the restriction of the Tamagawa measure on $\mathrm{Mat}_{n\times l}(\mathbb{A})$. The latter is by definition $\lambda^{n\times l}$ divided by the constant $\lambda(\mathbb{A}/K)^{nl}$, so we obtain

$$\int\limits_{\mathrm{Mat}_{n\times l}^{\mathrm{rk}=l}(\mathbb{A})} \Phi \, d\tau_{G/H} = \Delta^{-nl/2} \int\limits_{\mathrm{Mat}_{n\times l}(\mathbb{A})} \Phi \, d\lambda^{n\times l}$$

as required. □

3.3 Arakelov vector bundles

Definition 3.3.1 *An Arakelov (vector) bundle $\mathcal{E}$ over our 'arithmetic curve' $X = \mathrm{Spec}(\mathcal{O}_K) \cup X_\infty$ is a finitely generated projective $\mathcal{O}_K$-module $\mathcal{E}_{\mathcal{O}_K}$ endowed with*

- *a euclidean scalar product $\langle \_ , \_ \rangle_{\mathcal{E},v}$ on the real vector space $\mathcal{E}_{K_v}$ for every real place $v \in X_\infty$ and*
- *an hermitian scalar product $\langle \_ , \_ \rangle_{\mathcal{E},v}$ on the complex vector space $\mathcal{E}_{K_v}$ for every complex place $v \in X_\infty$*

where $\mathcal{E}_A := \mathcal{E}_{O_K} \otimes A$ for every $\mathcal{O}_K$-algebra A.

We say that $\mathcal{E}'$ is a subbundle of $\mathcal{E}$ and write $\mathcal{E}' \subseteq \mathcal{E}$ if $\mathcal{E}'_{\mathcal{O}_K}$ is a direct summand in $\mathcal{E}_{\mathcal{O}_K}$ and the scalar product on $\mathcal{E}'_{K_v}$ is the restriction of the one on $\mathcal{E}_{K_v}$ for every infinite place v. Hence every sub-vector space of $\mathcal{E}_K$ is the generic fibre of one and only one subbundle of $\mathcal{E}$.

From the data belonging to an Arakelov bundle $\mathcal{E}$, we can define a map

$$\| \,.\, \|_{\mathcal{E},v} : \mathcal{E}_{K_v} \longrightarrow \mathbb{R}^{\geq 0}$$

for every place $v \in X$:

- If v is finite, let $\|e\|_{\mathcal{E},v}$ be the minimum of the valuations $|a|_v$ of those elements $a \in K_v$ for which e lies in the subset $a \cdot \mathcal{E}_{\mathcal{O}_v}$ of $\mathcal{E}_{K_v}$. This is the nonarchimedean norm corresponding to $\mathcal{E}_{\mathcal{O}_v}$.
- If v is real, we put $\|e\|_{\mathcal{E},v} := \sqrt{\langle e, e \rangle_v}$, so we just take the norm coming from the given scalar product.
- If v is complex, we put $\|e\|_{\mathcal{E},v} := \langle e, e \rangle_v$ which is the square of the norm coming from our hermitian scalar product.

Recall that this is used in the definition of the Arakelov degree: If $\mathcal{L}$ is an Arakelov line bundle and $0 \neq s \in \mathcal{L}_K$ a nonzero generic section, then

$$\deg(\mathcal{L}) := -\log \prod_{v \in X} \|s\|_{\mathcal{L},v}$$

and the degree of an Arakelov vector bundle $\mathcal{E}$ is by definition the degree of the Arakelov line bundle $\det(\mathcal{E})$. $\mu(\mathcal{E}) := \deg(\mathcal{E})/\mathrm{rk}(\mathcal{E})$ is called the slope of $\mathcal{E}$. One can form the tensor product of two Arakelov bundles in a natural manner, and it has the property $\mu(\mathcal{E} \otimes \mathcal{F}) = \mu(\mathcal{E}) + \mu(\mathcal{F})$.

For each finite place v, we have

$$\mathcal{E}_{\mathcal{O}_v} = \{e \in \mathcal{E}_{K_v} : \|e\|_{\mathcal{E},v} \leq 1\}.$$

Slightly abusing notation, we take this equation as a definition of $\mathcal{E}_{\mathcal{O}_v}$ for infinite places v. Then

$$\mathcal{E}_{\mathcal{O}_\mathbb{A}} := \prod_{v \in X} \mathcal{E}_{\mathcal{O}_v}$$

is a compact neighborhood of 0 in the topological abelian group $\mathcal{E}_\mathbb{A}$.

Again carrying over from section 3.1, we get a canonical Haar measure on $\mathcal{E}_\mathbb{A}$

$$\lambda_\mathcal{E} := \prod_{v \in X} \lambda_{\mathcal{E},v}$$

whose local components are defined as follows:

- If $v \in X$ is a finite place, we normalize the Haar measure $\lambda_{\mathcal{E},v}$ on $\mathcal{E}_{K_v}$ by $\lambda_{\mathcal{E},v}(\mathcal{E}_{\mathcal{O}_v}) = 1$.
- If $v \in X_\infty$ is an infinite place, we choose an isometry of $\mathcal{E}_{K_v}$ onto K_v^n with the standard scalar product and let $\lambda_{\mathcal{E},v}$ be the pullback of the product measure λ_v^n from section 3.1.

Lemma 3.3.2 *Let $\mathcal{E}$ be an Arakelov vector bundle of rank n over X.*

i) The measure of the compact subset $\mathcal{E}_{\mathcal{O}_\mathbb{A}} \subseteq \mathcal{E}_\mathbb{A}$ depends only on the rank of $\mathcal{E}$. More precisely, $\lambda_\mathcal{E}(\mathcal{E}_{\mathcal{O}_\mathbb{A}}) = V(n)$ with

$$\begin{aligned} V(n) &= 2^{nr_2}\pi^{nd/2}(n/2)!^{-r_1}n!^{-r_2} \\ &= (2\pi e/n)^{nd/2} \cdot \mathrm{O}\left(n^{-(r_1+r_2)/2}\right) \quad \text{for } n \to \infty. \end{aligned}$$

ii) If $\phi_\mathbb{A} : \mathbb{A}^n \xrightarrow{\sim} \mathcal{E}_\mathbb{A}$ comes from a linear isomorphism $\phi : K^n \xrightarrow{\sim} \mathcal{E}_K$, then

$$\begin{aligned} \lambda_\mathcal{E} &= \exp(-\deg(\mathcal{E})) \cdot \phi_{\mathbb{A},*}\lambda^n \quad \textit{and in particular} \\ \lambda_\mathcal{E}(\mathcal{E}_\mathbb{A}/\mathcal{E}_K) &= \exp(-\deg(\mathcal{E})) \cdot \Delta^{n/2}. \end{aligned}$$

Proof: For i, note that $\lambda_{\mathcal{E},v}(\mathcal{E}_{\mathcal{O}_v}) = 1$ by definition if v is finite. If v is real,

$$\lambda_{\mathcal{E},v}(\mathcal{E}_{\mathcal{O}_v}) = V_\mathbb{R}(n) = \frac{\pi^{n/2}}{(n/2)!}$$

is the volume $V_\mathbb{R}(n)$ of the unit ball in $\mathbb{R}^n$, and if v is complex,

$$\lambda_{\mathcal{E},v}(\mathcal{E}_{\mathcal{O}_v}) = 2^n \cdot V_\mathbb{R}(2n) = \frac{(2\pi)^n}{n!}.$$

Multiplying these for all $v \in X$ gives the first formula. To obtain the asymptotic expression, just apply Stirling's formula to the factorials.

For ii, we compute the degree of $\mathcal{E}$ using the generic section $s \in \det \mathcal{E}_K$ which is the image of the canonical generator of $\det(K^n)$ under ϕ. More or less by definition of $\lambda_{\mathcal{E},v}$, it satisfies the relation

$$\lambda_{\mathcal{E},v} = \|s\|_{\det \mathcal{E},v} \cdot \phi_{v,*}\lambda_v^n$$

where of course $\phi_v : K_v^n \longrightarrow \mathcal{E}_{K_v}$ is the map induced by ϕ. The first claim of ii is the product of these relations. The last statement follows from this since $\lambda(\mathbb{A}/K) = \Delta^{1/2}$. □

3.4 No global sections

Recall that the global sections of an Arakelov bundle $\mathcal{E}$ over $X = \mathrm{Spec}(\mathcal{O}_K) \cup X_\infty$ are by definition the elements of the finite set

$$\Gamma(\mathcal{E}) := \mathcal{E}_K \cap \mathcal{E}_{\mathcal{O}_\mathbb{A}} \subseteq \mathcal{E}_\mathbb{A}.$$

This section deals with the following question: How big can the degree of an Arakelov bundle without nonzero global sections be?

Note that in the special case $K = \mathbb{Q}$, this corresponds to (lattice) sphere packings: In that case, an Arakelov bundle $\mathcal{E}$ is just a $\mathbb{Z}$-lattice in a euclidean real vector space, and $\Gamma(\mathcal{E}) = 0$

means that the (closed) balls of radius 1/2 centered at the lattice points are disjoint. Larger degree corresponds to denser packings.

The famous Minkowski-Hlawka theorem asserts that there are sphere packings of a certain density; it is proved in a nonconstructive way by averaging over the space of all lattices. In what follows, analogous arguments will prove the existence of Arakelov bundles of certain degrees without global sections.

To construct spaces of all Arakelov bundles $\mathcal{E}$, remember that $\mathcal{E}_K$ is a K-lattice in the free $\mathbb{A}$-module $\mathcal{E}_A$, i. e. a sub-vector space for which the map $\mathcal{E}_K \otimes_K \mathbb{A} \longrightarrow \mathcal{E}_{\mathbb{A}}$ is bijective. Conversely, if we fix the $\mathbb{A}$-module $\mathcal{E}_{\mathbb{A}}$ and its subset $\mathcal{E}_{\mathcal{O}_{\mathbb{A}}}$, then every K-lattice in $\mathcal{E}_{\mathbb{A}}$ is the generic fibre of a unique Arakelov bundle $\mathcal{E}'$ with $\mathcal{E}'_{\mathbb{A}} = \mathcal{E}_{\mathbb{A}}$ and $\mathcal{E}'_{\mathcal{O}_{\mathbb{A}}} = \mathcal{E}_{\mathcal{O}_{\mathbb{A}}}$. Hence the group $\mathrm{Gl}(\mathcal{E}_{\mathbb{A}})$ acts on the set of these bundles by the formula $(a\mathcal{E}')_K := a(\mathcal{E}'_K)$, and we get the usual identification of

$$\mathrm{Sl}(\mathcal{E}_{\mathbb{A}})/\mathrm{Sl}(\mathcal{E}_K)$$

with the set of such bundles having the same determinant as $\mathcal{E}$.

Proposition 3.4.1 *Let $\mathcal{E}$ be an Arakelov vector bundle of rank $n > 1$ over X.*

i) There is a unique invariant probability measure $\tau_{\mathcal{E}}$ on $\mathrm{Sl}(\mathcal{E}_{\mathbb{A}})/\mathrm{Sl}(\mathcal{E}_K)$.

ii) For every integrable function Φ on $\mathcal{E}_{\mathbb{A}}$, we have the equation

$$\int_{\mathrm{Sl}(\mathcal{E}_{\mathbb{A}})/\mathrm{Sl}(\mathcal{E}_K)} d\tau_{\mathcal{E}}(a) \sum_{0 \neq s \in a\mathcal{E}_K} \Phi(s) = \Delta^{-n/2} \cdot \exp\deg(\mathcal{E}) \int_{\mathcal{E}_{\mathbb{A}}} \Phi \, d\lambda_{\mathcal{E}}.$$

Proof: The uniqueness of $\tau_{\mathcal{E}}$ is clear. For the existence, choose a K-linear isomorphism $\phi : K^n \xrightarrow{\sim} \mathcal{E}_K$ and let

$$c(\phi) : \mathrm{Sl}_n(\mathbb{A})/\mathrm{Sl}_n(K) \xrightarrow{\sim} \mathrm{Sl}(\mathcal{E}_{\mathbb{A}})/\mathrm{Sl}(\mathcal{E}_K)$$

be the conjugation with $\phi_{\mathbb{A}} : \mathbb{A}^n \xrightarrow{\sim} \mathcal{E}_{\mathbb{A}}$. Then $\tau_{\mathcal{E}} := c(\phi)_*\tau$ proves i. For ii, we use the special case $l = 1$ of the mean value formula 3.2.1. If we identify $\mathbb{A}^n$ with $\mathcal{E}_{\mathbb{A}}$ by means of $\phi_{\mathbb{A}}$, we get

$$\int_{\mathrm{Sl}(\mathcal{E}_{\mathbb{A}})/\mathrm{Sl}(\mathcal{E}_K)} d\tau_{\mathcal{E}}(a) \sum_{0 \neq s \in a\mathcal{E}_K} \Phi(s) = \Delta^{-n/2} \int_{\mathbb{A}^n} \Phi \, d(\phi_{\mathbb{A},*}\lambda^n)$$

which implies the proposition because of Lemma 3.3.2.ii. □

Corollary 3.4.2 *Let $n > 1$ be an integer and assume that the real number μ satisfies*

$$\mu < -\frac{1}{n}\log V(n) + \log\sqrt{\Delta} = \frac{d}{2}(\log n - 1) + \log\sqrt{\frac{\Delta}{(2\pi)^d}} + \mathrm{O}\left(\frac{\log n}{n}\right).$$

Then there is an Arakelov bundle $\mathcal{E}$ over X with

$$\mathrm{rk}(\mathcal{E}) = n, \quad \mu(\mathcal{E}) = \mu \quad \text{and} \quad \Gamma(\mathcal{E}) = 0.$$

Proof: We compute the average number of global sections with the mean value formula just given. Choosing a bundle $\mathcal{E}$ with rank $n > 1$ and slope μ, we get

$$\int_{\mathrm{Sl}(\mathcal{E}_{\mathbb{A}})/\mathrm{Sl}(\mathcal{E}_K)} \operatorname{card}\left(\Gamma(a\mathcal{E}) \setminus 0\right) d\tau_{\mathcal{E}}(a) = \Delta^{-n/2} \cdot \exp(n\mu) \cdot V(n).$$

If μ satisfies the assumption, the right hand side of this equation is less than one and hence there is an $a \in \mathrm{Sl}(\mathcal{E}_{\mathbb{A}})$ such that the Arakelov bundle $a\mathcal{E}$ has no nonzero global section. □

Remarks 3.4.3 i) This corollary can be improved slightly, e. g. by taking into account that the number of nonzero global sections is always a multiple of the number of roots of unity in K. This improves the given bound by a summand proportional to $1/n$. However, even in the special case $K = \mathbb{Q}$ of sphere packings, according to [CoSl93] every known improvement of the corollary is by summands tending to zero for $n \to \infty$.

ii) On the other hand, we have the following upper bound for the slope of bundles without global sections. (At least in the special case $K = \mathbb{Q}$, several people have improved it by constants, i. e. by summands converging to a positive limit for $n \to \infty$. For an overview, see [CoSl93].)

Proposition 3.4.4 *Every Arakelov bundle $\mathcal{E}$ over X having rank $n > 0$ and slope*

$$\mu(\mathcal{E}) > -\frac{1}{n}\log V(n) + \log\left(2^d\sqrt{\Delta}\right) = \frac{d}{2}(\log n - 1) + \log\sqrt{\frac{2^d\Delta}{\pi^d}} + \mathrm{O}\left(\frac{\log n}{n}\right)$$

has a nonzero global section.

Proof: Because of Lemma 3.3.2, the assumption on the slope is equivalent to

$$\lambda_{\mathcal{E}}(\mathcal{E}_{\mathcal{O}_{\mathbb{A}}}) > 2^{nd}\lambda_{\mathcal{E}}(\mathcal{E}_{\mathbb{A}}/\mathcal{E}_K).$$

Taking into account the adelic version [Thu96, Theorem 3] of Minkowski's theorem on lattice points in convex sets, this implies $\Gamma(\mathcal{E}) \neq 0$. □

Now recall the notion of stability. For $1 \leq l \leq \mathrm{rk}(\mathcal{E})$, denote by $\mu^{(l)}_{\max}$ the supremum (in fact it is the maximum) of the slopes $\mu(\mathcal{E}')$ of subbundles $\mathcal{E}' \subseteq \mathcal{E}$ of rank l. $\mathcal{E}$ is said to be stable if $\mu^{(l)}_{\max} < \mu(\mathcal{E})$ holds for all $l < \mathrm{rk}(\mathcal{E})$, and semistable if $\mu^{(l)}_{\max} \leq \mu(\mathcal{E})$ for all l.

Theorem 3.4.5 *Let an Arakelov bundle $\mathcal{E}$ of rank $n_{\mathcal{E}} > 0$ over the arithmetic curve X be given. If $n_{\mathcal{F}}$ is a sufficiently large integer and $\mathcal{L}$ is an Arakelov line bundle satisfying*

$$\mu^{(l)}_{\max}(\mathcal{E}) + \frac{\deg(\mathcal{L})}{n_{\mathcal{F}}} \leq \frac{d}{2}(\log(l n_{\mathcal{F}}) - 1) + \log\sqrt{\frac{\Delta}{(2\pi)^d}} \tag{3.1}$$

for all $1 \leq l \leq n_{\mathcal{E}}$, then there is an Arakelov bundle $\mathcal{F}$ of rank $n_{\mathcal{F}}$ and determinant $\mathcal{L}$ such that

$$\Gamma(\mathcal{E} \otimes \mathcal{F}) = 0.$$

Corollary 3.4.6 *If $\mathcal{E}$ is semistable of rank $n_{\mathcal{E}}$, then for every sufficiently large integer $n_{\mathcal{F}}$ there is an Arakelov bundle $\mathcal{F}$ of rank $n_{\mathcal{F}}$ satisfying*

$$\mu(\mathcal{E} \otimes \mathcal{F}) = \frac{d}{2}(\log n_{\mathcal{F}} - 1) + \log\sqrt{\frac{\Delta}{(2\pi)^d}} \quad \text{and} \quad \Gamma(\mathcal{E} \otimes \mathcal{F}) = 0.$$

Proof: The method for proving this theorem is the same as for Corollary 3.4.2: We use the mean value formula to show that the average number of global sections is less than one. However, the argument is more involved here because our mean value formula deals only with matrices of full rank.

Let $\mathcal{F}$ be an arbitrary Arakelov bundle of rank $n_\mathcal{F} > n_\mathcal{E}$. Choose linear isomorphisms

$$\phi_\mathcal{E} : K^{n_\mathcal{E}} \xrightarrow{\sim} \mathcal{E}_K \quad \text{and} \quad \phi_\mathcal{F} : K^{n_\mathcal{F}} \xrightarrow{\sim} \mathcal{F}_K$$

and denote by

$$\phi_\mathbb{A} : \mathrm{Mat}_{n_\mathcal{F} \times n_\mathcal{E}}(\mathbb{A}) \xrightarrow{\sim} (\mathcal{E} \otimes \mathcal{F})_\mathbb{A}$$

the map induced by their tensor product. Using these to identify, the mean value formula 3.2.1 takes the form

$$\int\limits_{\mathrm{Sl}(\mathcal{F}_\mathbb{A})/\mathrm{Sl}(\mathcal{F}_K)} d\tau_\mathcal{F}(a) \sum_{\substack{s \in (\mathcal{E} \otimes a\mathcal{F})_K \\ s \notin (\mathcal{E}' \otimes a\mathcal{F})_K \text{ for all } \mathcal{E}' \subsetneq \mathcal{E}}} \Phi(s) = \Delta^{-n_\mathcal{E} n_\mathcal{F}/2} \int\limits_{(\mathcal{E} \otimes \mathcal{F})_\mathbb{A}} \Phi \, d(\phi_{\mathbb{A},*} \lambda^{n_\mathcal{F} \times n_\mathcal{E}}).$$

Now we take for Φ the indicator function of $(\mathcal{E} \otimes \mathcal{F})_{\mathcal{O}_\mathbb{A}}$, use Lemma 3.3.2 to evaluate the right hand side and get

$$\int\limits_{\mathrm{Sl}(\mathcal{F}_\mathbb{A})/\mathrm{Sl}(\mathcal{F}_K)} \mathrm{card}\left(\Gamma(\mathcal{E} \otimes a\mathcal{F}) \setminus \bigcup_{\mathcal{E}' \subsetneq \mathcal{E}} \Gamma(\mathcal{E}' \otimes a\mathcal{F})\right) d\tau_\mathcal{F}(a) = \frac{V(n_\mathcal{E} n_\mathcal{F})}{\Delta^{n_\mathcal{E} n_\mathcal{F}/2}} \exp\deg(\mathcal{E} \otimes \mathcal{F}).$$

In order to take care of *all* global sections, note that for every $s \in \Gamma(\mathcal{E} \otimes a\mathcal{F})$ there is a unique minimal subbundle $\mathcal{E}' \subseteq \mathcal{E}$ with $s \in \Gamma(\mathcal{E}' \otimes a\mathcal{F})$: it is the subbundle whose generic fibre is the image of the map $s_K : (a\mathcal{F}_K)^{\mathrm{dual}} \longrightarrow \mathcal{E}_K$ induced by s. So we perform a summation over all these subbundles $\mathcal{E}'$. The arguments just given for $\mathcal{E}$ apply to $\mathcal{E}'$ as well, and we obtain

$$\int\limits_{\mathrm{Sl}(\mathcal{F}_\mathbb{A})/\mathrm{Sl}(\mathcal{F}_K)} \mathrm{card}\left(\Gamma(\mathcal{E} \otimes a\mathcal{F}) \setminus 0\right) d\tau_\mathcal{F}(a) = \sum_{l=1}^{n_\mathcal{E}} \frac{V(l n_\mathcal{F})}{\Delta^{l n_\mathcal{F}/2}} \zeta_\mathcal{E}^{(l)}(n_\mathcal{F}) \cdot \exp(l \deg(\mathcal{F})) \tag{3.2}$$

using the Zeta function $\zeta_\mathcal{E}^{(l)}$ that is defined by

$$\zeta_\mathcal{E}^{(l)}(s) = \sum_{\substack{\mathcal{E}' \subseteq \mathcal{E} \\ \mathrm{rk}(\mathcal{E}') = l}} \exp(s \cdot \deg(\mathcal{E}')). \tag{3.3}$$

Remark 3.4.7 In [BaMa90] Batyrev and Manin have introduced the Zeta function of a projective variety over K endowed with a metrized line bundle. In their language, $\zeta_\mathcal{E}^{(l)}$ is the Zeta function of the Grassmannian attached to $\mathcal{E}$.

Lemma 3.4.8 *There is a constant* $C = C(\mathcal{E})$ *such that*

$$\zeta_\mathcal{E}^{(l)}(s) \leq C \cdot \exp(s \cdot l \mu_{\max}^{(l)}(\mathcal{E}))$$

for all sufficiently large real numbers s *and all* $1 \leq l \leq n_\mathcal{E} = \mathrm{rk}(\mathcal{E})$.

Proof: For each real number T, denote by $N_{\mathcal{E}}^{(l)}(T)$ the number of subbundles $\mathcal{E}' \subseteq \mathcal{E}$ of rank l and degree at least $-T$. It will be shown that there are real constants C_1 and C_2 (which are allowed to depend on $\mathcal{E}$) such that

$$N_{\mathcal{E}}^{(l)}(T) \leq \exp(C_1 T + C_2) \tag{3.4}$$

for all T. This will imply the lemma as follows:

If we order the summands in (3.3) according to their magnitude, we get

$$\begin{aligned}\zeta_{\mathcal{E}}^{(l)}(s) &\leq \sum_{\nu=0}^{\infty} N_{\mathcal{E}}^{(l)}\left(-l\mu_{\max}^{(l)}(\mathcal{E}) + \nu + 1\right) \cdot \exp\left(s \cdot (l\mu_{\max}^{(l)}(\mathcal{E}) - \nu)\right) \\ &\leq \exp(s \cdot l\mu_{\max}^{(l)}(\mathcal{E})) \cdot \sum_{\nu=0}^{\infty} \frac{C_3}{\exp((s - C_1)\nu)}\end{aligned}$$

using (3.4). But the last sum is a convergent geometric series for all $s > C_1$ and decreases as s grows, hence the sum is bounded for $s \geq C_1 + 1$ and the lemma is proved up to (3.4).

Several people have given more precise asymptotic formulas for the number of points of bounded height on projective spaces or Grassmannians over K from which (3.4) could be deduced. See for example [Sch79], [FMT89], [Thu92] or [Gas99]. However the statement (3.4) needed here is so weak that the following rough form of the argument will do:

Every subbundle $\mathcal{E}' \subseteq \mathcal{E}$ of rank l gives us a line bundle $\det(\mathcal{E}')$ in the exterior power $\bigwedge^l \mathcal{E}$ from which it can be reconstructed, hence

$$N_{\mathcal{E}}^{(l)}(T) \leq N_{\bigwedge^l \mathcal{E}}^{(1)}(T).$$

So without loss of generality, we can restrict ourselves to counting line bundles in $\mathcal{E}$. Because the class number of O_K is finite, there is a constant C_4 depending only on K such that every Arakelov line bundle $\mathcal{L}$ over X has a section $0 \neq s \in \mathcal{L}_{\mathcal{O}_K}$ for which the product over all *finite* places v of $\|s\|_{\mathcal{L},v}$ is at least $\exp(-C_4)$, which is equivalent to

$$\log \prod_{v \in X_\infty} \|s\|_{\mathcal{L},v} \leq C_4 - \deg(\mathcal{L}). \tag{3.5}$$

According to Dirichlet's Unit Theorem, the quotient of kernel modulo image in

$$\mathcal{O}_K^* \xrightarrow{\prod \log|\cdot|_v} \prod_{v \in X_\infty} \mathbb{R} \xrightarrow{+} \mathbb{R}$$

is compact, so (3.5) can be strengthened to

$$\log\langle s, s\rangle_{\mathcal{L},v} \leq C_5 - \deg(\mathcal{L})/d \quad \text{for all } v \in X_\infty \tag{3.6}$$

by multiplying s with an appropriate unit in $\mathcal{O}_K$. This shows that

$$N_{\mathcal{E}}^{(1)}(T) \leq N'_{\mathcal{E}}(C_5 + T/d)$$

where $N'_{\mathcal{E}}(T)$ denotes the number of nonzero sections $s \in \mathcal{E}_{\mathcal{O}_K}$ with $\log\langle s, s\rangle_{\mathcal{E},v} \leq T$ for all infinite places v. Since this is the number of points of the lattice

$$\mathcal{E}_{\mathcal{O}_K} \subseteq \prod_{v \in X_\infty} \mathcal{E}_{K_v}$$

in a homogeneously expanding domain, it grows linearly with the volume of that domain (for example by [Lan70, Chapter VI, Theorem 2]) which means that

$$N'_{\mathcal{E}}(T) \leq \exp(dn_{\mathcal{E}}T + C_6)$$

and completes the proof of the lemma. □

Now return to the proof of the theorem. We show that every integer $n_{\mathcal{F}} > n_{\mathcal{E}}$ which is so large that Lemma 3.4.8 holds for $s = n_{\mathcal{F}}$ and

$$V(n) \cdot C(\mathcal{E}) < \frac{1}{n_{\mathcal{E}}} \left(\frac{2\pi e}{n} \right)^{dn/2} \qquad \text{for all } n \geq n_{\mathcal{F}}$$

will do the trick. In fact, for every Arakelov bundle $\mathcal{F}$ of such a big rank $n_{\mathcal{F}}$, the average number of global sections

$$\int\limits_{\mathrm{Sl}(\mathcal{F}_{\mathbb{A}})/\mathrm{Sl}(\mathcal{F}_K)} \mathrm{card}\left(\Gamma(\mathcal{E} \otimes a\mathcal{F}) \setminus 0\right) \, d\tau_{\mathcal{F}}(a)$$

is smaller than

$$\frac{1}{n_{\mathcal{E}}} \sum_{l=1}^{n_{\mathcal{E}}} \Delta^{-ln_{\mathcal{F}}/2} \cdot \left(\frac{2\pi e}{ln_{\mathcal{F}}} \right)^{dln_{\mathcal{F}}/2} \cdot \exp\left(ln_{\mathcal{F}}(\mu^{(l)}_{\max}(\mathcal{E}) + \mu(\mathcal{F})) \right) \tag{3.7}$$

because of (3.2). Now let an Arakelov line bundle $\mathcal{L}$ satisfying the condition (3.1) of the theorem be given and choose $\mathcal{F}$ as above with determinant $\mathcal{L}$. Then every summand of (3.7) is at most one, and hence

$$\int\limits_{\mathrm{Sl}(\mathcal{F}_{\mathbb{A}})/\mathrm{Sl}(\mathcal{F}_K)} \mathrm{card}\left(\Gamma(\mathcal{E} \otimes a\mathcal{F}) \setminus 0\right) \, d\tau_{\mathcal{F}}(a) \quad < \quad 1$$

which shows that there is an $a \in \mathrm{Sl}(\mathcal{F}_{\mathbb{A}})$ such that $\Gamma(\mathcal{E} \otimes a\mathcal{F}) = 0$. □

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