Modulräume von Bündeln - Rationalität, Existenz von Poincaré-Familien und Geradenbündel auf den Modulstacks

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Summary

Moduli spaces are a central subject in modern algebraic geometry. They originate from fundamental classification problems, like the classification of algebraic varieties, or the classification of vector bundles on a fixed variety. It turns out that such objects usually have some discrete invariants. Fixing them, the isomorphism classes of these objects tend to form an algebraic variety, their moduli space.

This habilitation thesis deals with moduli spaces of vector bundles, or more generally of principal bundles, over a fixed algebraic variety. It consists of the following papers:

[1] N. Hoffmann. The Moduli Stack of Vector Bundles on a Curve. In I. Biswas, R.S. Kulkarni, and S. Mitra, editors, *Teichmüller Theory and Moduli Problem (Allahabad 2006)*, pp. 387–394. Ramanujan Math. Soc. Lecture Notes 10, 2010.

[2] N. Hoffmann. Moduli stacks of vector bundles on curves and the King-Schofield rationality proof. In F. Bogomolov and Y. Tschinkel, editors, *Cohomological and Geometric Approaches to Rationality Problems. New Perspectives*, pages 133–148. Progress in Mathematics 282, Birkhäuser, 2010.

[3] N. Hoffmann. Rationality and Poincaré families for vector bundles with extra structure on a curve. *Int. Math. Res. Not., Article ID rnm010, 30 p.*, 2007.

[4] N. Hoffmann. On Moduli Stacks of G-bundles over a Curve. In A. Schmitt, editor, *Affine Flag Manifolds and Principal Bundles (Berlin 2008).* to appear in Trends in Mathematics, Birkhäuser, 2010.

[5] I. Biswas and N. Hoffmann. Poincaré families and automorphisms of principal bundles on a curve. *C. R., Math., Acad. Sci. Paris*, 347(21-22):1285–1288, 2009.

[6] I. Biswas and N. Hoffmann. Some moduli stacks of symplectic bundles on a curve are rational. *Adv. Math.*, 219:1150–1176, 2008.

[7] I. Biswas and N. Hoffmann. The line bundles on moduli stacks of principal bundles on a curve. *Documenta Math.*, 15:35–72, 2010.

[8] I. Biswas and N. Hoffmann. Poincaré families of G-bundles on a curve. preprint arXiv:1001.2123 (submitted). available at `http://www.arXiv.org`.

[9] N. Hoffmann. The moduli space of special instanton bundles is rational.

The texts [1, 2, 3] treat moduli spaces of vector bundles on C, a smooth projective curve over an algebraically closed field k of arbitrary characteristic. Then [4, 5, 6, 7, 8] are more generally about moduli spaces of principal bundles on C. Finally, [9] deals with vector bundles on higher-dimensional complex projective spaces $\mathbb{P}^{2n+1}$.

Moduli spaces parameterize the isomorphism classes of algebro-geometric objects, but they are also interesting examples of algebraic varieties themselves. Thus it is natural to ask for their basic invariants as varieties, like their birational type. The moduli spaces considered here tend to be unirational, so the interesting question is whether they are actually rational.

Vector bundles or principal bundles usually have nontrivial automorphisms. This implies, by a standard argument, that their moduli functor is usually not representable (in fact not even a sheaf). So their moduli spaces are only coarse moduli schemes. One way to measure this defect is to ask whether there is a family of bundles, parameterized by the coarse moduli scheme (or some dense open part of it), whose restriction to every point in the moduli space belongs to the isomorphism class given by that point. Such a family is called a *Poincaré family* here, since it generalizes the notion of a Poincaré bundle on the product of an abelian variety with its dual. (Such a family is sometimes also called a universal family, although it need not be unique. If it exists, the moduli scheme is also said to be fine, since it then represents some slight modification of the original moduli functor.)

These two classical questions about coarse moduli schemes are studied in the present work. The main technical innovation is the use of moduli stacks. In this context, they have two advantages:

- They help to clarify the relation between the birational type of the coarse moduli scheme and the existence of Poincaré families (for small open subschemes), since both are contained in the birational type of the moduli stack.
- The existence of Poincaré families for the coarse moduli scheme turns out to be closely related to the existence of line bundles on the moduli stack with certain properties. This actually allows to answer the question completely for principal bundles on the curve C; cf. [8].

Here is now a more detailed description of the individual papers; see also their respective introductions for more specific outlines of their content.

The expository text [1] explains briefly and not too technically the notions of stack and of algebraic stack, illustrating them with the example of the moduli stack of vector bundles on the curve C. Motivated by a comparison of gluing for vector bundles and for maps to a scheme, it is argued that a fine moduli space of vector bundles cannot be a set with some geometric structure, it should be a category with some geometric structure. Then stacks are introduced as some sort of sheaves of categories (more precisely of groupoids). Finally, the two algebraicity conditions for stacks by Deligne-Mumford [DM] and by Artin [Ar] are explained, as analogues of the condition for a sheaf of sets to be (representable by) a scheme.

The paper [2] treats the rationality question for moduli spaces of vector bundles with fixed determinant on the curve C, which is assumed to be of genus at least 2. Given a line bundle L on C, consider the coarse moduli scheme $\mathfrak{Bun}_{r,L}$ of stable vector bundles E on C with rank r and determinant $\Lambda^r E \cong L$. Following work of Tyurin [T1, T2] and Newstead [N1, N2], it has been believed for a long time that $\mathfrak{Bun}_{r,L}$ is rational if r and the degree of L are coprime; this was finally proved by King and Schofield [KS]. The proof starts by showing that $\mathfrak{Bun}_{r,L}$ is birational to a twisted Grassmannian bundle over $\mathfrak{Bun}_{r_1,L_1}$ for some $r_1 < r$; then induction on the rank is used. However, they need a stronger induction hypothesis, since r_1 and $\deg(L_1)$ may no longer be coprime; it involves some Brauer class ψ_{r_1,L_1} over $\mathfrak{Bun}_{r_1,L_1}$ in order to control the twist of the Grassmannian bundle. The point of

[2] is to simplify this proof conceptually by the use of moduli stacks. Following the strategy of King-Schofield, the Brauer class is replaced by the corresponding $\mathbb{G}_m$-gerbe, which is simply the moduli stack of the vector bundles E in question; thus one just has to keep track of the scalar automorphisms $\mathbb{G}_m \subseteq \mathrm{Aut}(E)$. The paper [2] contains a complete proof of the King-Schofield result in stack language, based on a birational study of Grassmannian bundles over $\mathbb{G}_m$-gerbes. An appendix summarizes the properties of moduli stacks of vector bundles that are needed.

This clarification of the rationality proof for vector bundles has made the systematic generalization [3] to vector bundles with extra structure possible. These include, for example, vector bundles with parabolic structures [MS, Bi], stable pairs [Br, Th], coherent systems [LP, RV, KN], and more generally decorated vector bundles [Sch]. Using again vector bundles and Grassmannian bundles over $\mathbb{G}_m$-gerbes, it is proved that the moduli stacks in question are birational to an affine space times moduli stacks of vector bundles without extra structure, of (usually) smaller rank. Besides rationality of some of the coarse moduli schemes, this also shows which of them admit Poincaré families (on small open subschemes), since this is known for vector bundles due to Ramanan [Ra]. In fact the obstruction against such Poincaré families is precisely the Brauer class ψ used by King and Schofield. Like for vector bundles without extra structures, rationality is proved here in those cases where Poincaré families exist. The other cases reduce to moduli spaces of vector bundles with trivial determinant on C, where the rationality problem is still wide open.

The next papers [4, 5, 6, 7, 8] deal with generalizations of these results from vector bundles to principal bundles on the curve C. The mainly expository text [4] explains some basic properties of the moduli stack $\mathcal{M}_G$ of principal G-bundles on C for linear algebraic groups G over k, like algebraicity and smoothness. Its main result is that for smooth connected reductive G, the connected components $\mathcal{M}_G^d$ of $\mathcal{M}_G$ are indexed by elements $d \in \pi_1(G)$, which is by definition the quotient of the abelian group $\mathrm{Hom}(\mathbb{G}_m, T_G)$ for a maximal torus $T_G \subseteq G$ modulo the subgroup generated by the coroots of G. (This description of $\pi_0(\mathcal{M}_G)$ is well-established folklore, but there seemed to be no published reference for it in full generality, covering also the case of positive characteristic.)

Let $\mathfrak{M}_G^{d,\mathrm{s}}$ denote the coarse moduli scheme[1] of stable principal G-bundles of type $d \in \pi_1(G)$. The short note [5] gives a necessary condition for the existence of Poincaré families on arbitrarily small open subschemes $U \subseteq \mathfrak{M}_G^{d,\mathrm{s}}$, namely that every character $Z_G \to \mathbb{G}_m$ of the center $Z_G \subseteq G$ can be extended to the automorphism group scheme $\mathrm{Aut}(E)$ for every principal G-bundle E of the given type $d \in \pi_1(G)$. The proof consists of a simple argument using line bundles on the stack $\mathcal{M}_G^d$. This criterion allows to reprove the earlier results of Ramanan [Ra] and Balaji-Biswas-Nagaraj-Newstead [BBNN]. It also allows to decide the question at hand for the moduli spaces of orthogonal and of symplectic bundles. In the case of twisted symplectic bundles, i. e., of rank $2n$ vector bundles E endowed with a symplectic form $b : E \otimes E \to L$ with values in a fixed line bundle L, the result is that there exist Poincaré families if and only if n and $\deg(L)$ are both odd.

The following paper [6] treats the rationality problem for moduli of such twisted symplectic bundles $(E, b : E \otimes E \to L)$ on C. By analogy with the case of vector bundles, the result on Poincaré families suggests that they could be rational if their fixed discrete parameters $n = \mathrm{rank}(E)/2$ and $\deg(L)$ are both odd. This rationality statement is proved in [6]. Actually it is proved that the moduli stack is birational to an affine space times $B\mathbb{G}_m$, which would be impossible if there were no Poincaré families. The proof starts by finding a 'canonical' line subbundle in E for every sufficiently general twisted symplectic bundle $(E, b : E \otimes E \to L)$ with n and $\deg(L)$ odd. This allows to reconstruct E from a bundle of lower rank together

[1] In [5], this coarse moduli scheme is denoted by $M^{\mathrm{s}}_{G,d}$.

with some appropriate extension data; these can all be parameterized rationally.

As the arguments in [5] indicate, the information about Poincaré families can be obtained from the Picard group of the moduli stacks $\mathcal{M}_G^d$. The paper [7] determines all line bundles on these moduli stacks $\mathcal{M}_G^d$, for all connected reductive groups G over algebraically closed fields k of arbitrary characteristic. This generalizes earlier results of Kumar-Narasimhan-Ramanathan [KNR, KN] and Beauville-Laszlo-Sorger [BLS, LS, So] for simply connected or classical groups over $k = \mathbb{C}$. It gives in particular an algebraic proof for Teleman's [Te] result on semisimple groups over $\mathbb{C}$, which he proves using topological and analytic methods. The main tools are Faltings' [Fa] work for simply connected groups in arbitrary characteristic, and Laszlo's [La] method of descent along torsors under group stacks.

Due to its generality, the main result of [7], namely the description of $\mathrm{Pic}(\mathcal{M}_G^d)$, is slightly long to state. Here are its main features:

- The abelian group of homomorphisms from $\pi_1(G)$ into the Jacobian J_C embeds canonically into $\mathrm{Pic}(\mathcal{M}_G^d)$.
- The quotient is a finitely generated free abelian group, denoted by $\mathrm{NS}(\mathcal{M}_G^d)$.
- If $G = T$ is a torus, with cocharacter lattice $\Lambda_T := \mathrm{Hom}(\mathbb{G}_m, T)$, then $\mathrm{NS}(\mathcal{M}_T^d)$ is the direct sum of $\mathrm{Hom}(\Lambda_T, \mathbb{Z})$ and the abelian group of bilinear maps $b : \Lambda_T \otimes \Lambda_T \to \mathrm{End}\, J_C$ satisfying $b(\lambda_1 \otimes \lambda_2)^\dagger = b(\lambda_2 \otimes \lambda_1)$ for the Rosati involution † on $\mathrm{End}\, J_C$.
- If G is semisimple, with maximal torus $T_G \subseteq G$, then $\mathrm{NS}(\mathcal{M}_G^d)$ is the abelian group of symmetric Weyl-invariant bilinear forms $b : \Lambda_{T_G} \otimes \Lambda_{T_G} \to \mathbb{Z}$ whose restriction to the coroot lattice is even.

For the proof, and also for the application below, it is important that the description of $\mathrm{Pic}(\mathcal{M}_G^d)$ is functorial in G, in particular with respect to the inclusion $\iota : T_G \hookrightarrow G$ of a maximal torus. If $\delta \in \pi_1(T_G) = \Lambda_{T_G}$ is a lift of $d \in \pi_1(G)$, then ι induces $\iota_* : \mathcal{M}_{T_G}^\delta \to \mathcal{M}_G^d$ by extension of the structure group. We have in particular:

- For G semisimple, $\iota^* : \mathrm{Pic}(\mathcal{M}_G^d) \to \mathrm{Pic}(\mathcal{M}_{T_G}^\delta)$ is compatible with the map

$$\iota^{\mathrm{NS},\delta} : \mathrm{NS}(\mathcal{M}_G^d) \longrightarrow \mathrm{NS}(\mathcal{M}_{T_G}^\delta), \qquad b \mapsto \big(b(-\delta \otimes \_),\ \mathrm{id}_{J_C} \cdot b\big).$$

These are the main ingredients to the general description of $\mathrm{Pic}(\mathcal{M}_G^d)$, and of its functoriality in G, proved in [7].

Using these results, the paper [8] gives a complete answer to the existence question for Poincaré families on the coarse moduli schemes $\mathfrak{M}_G^{d,\mathrm{s}}$ of stable principal G-bundles E over C. It answers, again in terms of bilinear forms on the root system of G, the following two variants of the question:

- Is there a Poincaré family parameterized by some (arbitrarily small) nonempty open subscheme $U \subseteq \mathfrak{M}_G^{d,\mathrm{s}}$?
- Is there a Poincaré family parameterized by the open locus $\mathfrak{M}_G^{d,\mathrm{rs}} \subseteq \mathfrak{M}_G^{d,\mathrm{s}}$ of regularly stable principal G-bundles?

Here a stable principal G-bundle E over C is called regularly stable if $\mathrm{Aut}(E) = Z_G$, the center of G. The regularly stable locus $\mathcal{M}_G^{d,\mathrm{rs}}$ in the moduli stack $\mathcal{M}_G^d$ is a gerbe with band Z_G over $\mathfrak{M}_G^{d,\mathrm{rs}}$; its class in $\mathrm{H}^2(\mathfrak{M}_G^{d,\mathrm{rs}}, Z_G)$ is the obstruction against Poincaré families. In fact, [8] determines the order (or 'period') of this obstruction class, and of its restriction to the generic point of $\mathfrak{M}_G^{d,\mathrm{rs}}$.

The method to do this is to translate the question into one about line bundles $\mathcal{L}$ on $\mathcal{M}_G^d$. Given such a line bundle $\mathcal{L}$ and a principal G-bundle E over C of the given type $d \in \pi_1(G)$, the group $\mathrm{Aut}(E)$ acts on the fiber of $\mathcal{L}$ over the moduli point of E. Thus we get a character of $\mathrm{Aut}(E)$. Its restriction to the subgroup $Z_G \subseteq \mathrm{Aut}(E)$ is independent of E, since $\mathcal{M}_G^d$ is connected and $\mathrm{Hom}(Z_G, \mathbb{G}_m)$ is discrete. Hence we obtain a homomorphism $\mathrm{Pic}(\mathcal{M}_G^d) \to \mathrm{Hom}(Z_G, \mathbb{G}_m)$. It turns out that this homomorphism factors through $\mathrm{NS}(\mathcal{M}_G^d)$, and even through $\mathrm{NS}(\mathcal{M}_{T_G}^\delta)$, where it can be described explicitly in the language of [7]. Using Giraud's [Gi] theory of gerbes, the knowledge of this homomorphism then answers the questions at hand.

However, these arguments only work if the regularly stable locus is non-empty. In characteristic 0, this was known for any genus $g_C \geq 2$. But in positive characteristic, it turned out to be more difficult than expected; it is proved in [8] for genus $g_C \geq 3$. In fact the statements about $\mathfrak{M}_G^{d,\mathrm{rs}}$ require to prove that the complement of $\mathcal{M}_G^{d,\mathrm{rs}}$ has codimension ≥ 2 in $\mathcal{M}_G^d$; this is proved in [8] for genus $g_C \geq 4$.

The paper [9] deals with instanton bundles on a complex projective space $\mathbb{P}^{2n+1}$. They are by definition algebraic vector bundles of rank $2n$ on $\mathbb{P}^{2n+1}$, satisfying certain conditions. These are motivated by the Penrose transformation, which in the case $n = 1$ yields a correspondence with solutions of certain Yang-Mills equations [AW, DV]; Okonek and Spindler [OS] generalized the definition to the case $n \geq 1$. Special instanton bundles have been introduced, and their moduli spaces studied, by Hirschowitz-Narasimhan [HN] for $n = 1$, and by Spindler-Trautmann [ST] for all $n \geq 1$. The main result of [9] is that the Spindler-Trautmann moduli spaces of special instanton bundles are all rational; this generalizes a result for $n = 1$ in [HN].

Like in the case of vector bundles on a curve, the proof involves the rationality of some Severi-Brauer varieties. Their Brauer classes are related to the existence of Poincaré families; Spindler and Trautmann determine in [ST] when these exist. A somewhat surprising feature of this situation is that the coarse moduli schemes are proved to be rational even in those cases where Poincaré families don't exist.

The main new ingredient in the proof, compared to the special case $n = 1$ in [HN], is the no-name lemma about vector spaces modulo linear group actions. It allows to reduce the problem to the rationality of some quotient modulo PGL_2, where the invariant ring is known explicitly. That's how rationality of the coarse moduli scheme is proved here, without at the same time constructing a Poincaré family for some open part of it. Although the rationality proof stays within the language of schemes, the stacky point of view has been helpful in finding it, and in clarifying the subtle existence questions for Poincaré families parameterized by some auxiliary moduli spaces in the Spindler-Trautmann construction; these issues are also explained in a series of remarks in [9].

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