THE BRAUER GROUP OF MODULI SPACES OF VECTOR BUNDLES OVER A REAL CURVE

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Abstract. Let X be a geometrically connected smooth projective curve of genus $g_X \geq 2$ over $\mathbb{R}$. Let $M(r, \xi)$ be the coarse moduli space of geometrically stable vector bundles E over X of rank r and determinant ξ, where ξ is a real point of the Picard variety $\underline{\mathrm{Pic}}^d(X)$. If $g_X = r = 2$, then let d be odd. We compute the Brauer group of $M(r, \xi)$.

1. Introduction

Let $X_{\mathbb{C}}$ be a connected smooth projective curve of genus $g_X \geq 2$ over $\mathbb{C}$. Fix integers $r \geq 2$ and d. Given a line bundle $\xi_{\mathbb{C}}$ of degree d over $X_{\mathbb{C}}$, we denote by $M(r, \xi_{\mathbb{C}})$ the coarse moduli space of stable vector bundles over $X_{\mathbb{C}}$ of rank r and determinant $\xi_{\mathbb{C}}$.

The Picard group of such moduli spaces has been studied intensively; see for example [DN, KN, LS, So, BLS, Fa, Te, BHo1]. We view the Brauer group as a natural higher order analogue of the Picard group. It is related to the classical rationality problem [CS].

We assume that d is odd if $g_X = r = 2$; otherwise d is arbitrary. The Brauer group of $M(r, \xi_{\mathbb{C}})$ has been computed in [BBGN]; the result is a canonical isomorphism

$$\mathrm{Br}(M(r, \xi_{\mathbb{C}})) \cong \mathbb{Z}/\mathrm{gcd}(r, d).$$

The corresponding generator $\beta_{\mathbb{C}} \in \mathrm{Br}(M(r, \xi_{\mathbb{C}}))$ can be viewed as the obstruction against the existence of a Poincaré bundle, or universal vector bundle, over $M(r, \xi_{\mathbb{C}}) \times X_{\mathbb{C}}$.

Now suppose $X_{\mathbb{C}} = X \otimes_{\mathbb{R}} \mathbb{C}$ for a smooth projective curve X over $\mathbb{R}$. Then some of the above moduli spaces carry interesting real algebraic structures, and there has been a growing interest in understanding these structures [BhB, BHH, BHu, Sch]. In this note, we compute the Brauer group of such real algebraic moduli spaces.

More precisely, assume that the line bundle $\xi_{\mathbb{C}}$ comes from a real point ξ of the Picard variety $\underline{\mathrm{Pic}}^d(X)$. Let $M(r, \xi)$ be the coarse moduli space of geometrically stable vector bundles E over X of rank r and determinant ξ. It is a smooth quasiprojective variety over $\mathbb{R}$, with $M(r, \xi) \otimes_{\mathbb{R}} \mathbb{C} \cong M(r, \xi_{\mathbb{C}})$; see Section 2. Our main result, Theorem 3.3, describes the Brauer group of $M(r, \xi)$ as follows.

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Theorem 1.1. *With $\chi := r(1 - g_X) + d$, there is a canonical isomorphism*

$$\mathrm{Br}(M(r,\xi)) \cong \begin{cases} \mathbb{Z}/\gcd(r,\chi) \oplus \mathbb{Z}/2 & \text{if } \xi \text{ comes from a line bundle defined over } \mathbb{R}, \\ \mathbb{Z}/\gcd(2r,\chi) & \text{otherwise.} \end{cases}$$

Note that $\gcd(r,\chi) = \gcd(r,d)$. The groups $\mathbb{Z}/\gcd(r,\chi)$ and $\mathbb{Z}/\gcd(2r,\chi)$ are generated by a canonical class $\beta \in \mathrm{Br}(M(r,\xi))$, the obstruction against a Poincaré bundle over $M(r,\xi) \times X$. The order of this obstruction class β is computed in Proposition 3.2. The remaining direct summand $\mathbb{Z}/2$ comes from the Brauer group of $\mathbb{R}$.

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2. Moduli of vector bundles over a real curve

Let X be a geometrically connected smooth projective algebraic curve of genus $g_X \geq 2$ defined over $\mathbb{R}$. We will denote the base change from $\mathbb{R}$ to $\mathbb{C}$ by a subscript $\mathbb{C}$. In particular, $X_\mathbb{C} := X \otimes_\mathbb{R} \mathbb{C}$ is the associated algebraic curve over $\mathbb{C}$.

Let $\sigma : \mathbb{C} \longrightarrow \mathbb{C}$ denote the complex conjugation. The involutive morphism of schemes

$$\sigma_X := \mathrm{id}_X \otimes \sigma : X_\mathbb{C} \longrightarrow X_\mathbb{C}$$

lies over $\sigma : \mathbb{C} \longrightarrow \mathbb{C}$. The closed points of $X_\mathbb{C}$ fixed by σ_X are the real points of X.

Let ξ be a real point of the Picard variety $\underline{\mathrm{Pic}}(X)$. Viewing the associated complex point $\xi_\mathbb{C}$ of $\underline{\mathrm{Pic}}(X_\mathbb{C})$ as a line bundle over $X_\mathbb{C}$, we have $\xi_\mathbb{C} \cong \sigma_X^*(\xi_\mathbb{C})$.

A *real* (respectively, *quaternionic*) structure on $\xi_\mathbb{C}$ is by definition an isomorphism

$$\eta : \xi_\mathbb{C} \longrightarrow \sigma_X^*(\xi_\mathbb{C})$$

of line bundles over $X_\mathbb{C}$ with $\sigma_X^*\eta \circ \eta = \mathrm{id}_{\xi_\mathbb{C}}$ (respectively, $\sigma_X^*\eta \circ \eta = -\mathrm{id}_{\xi_\mathbb{C}}$). The line bundle $\xi_\mathbb{C}$ admits either a real structure η or a quaternionic structure η, and in both cases the resulting pair $(\xi_\mathbb{C}, \eta)$ is uniquely determined up to an isomorphism; cf. for example [Ve, Proposition 2.5] or [BHH, Proposition 3.1].

The real point ξ of $\underline{\mathrm{Pic}}(X)$ is called *quaternionic* if $\xi_\mathbb{C}$ admits a quaternionic structure. Otherwise, $\xi_\mathbb{C}$ admits a real structure, so we can view ξ as a real line bundle over X.

A vector bundle E over X is called *geometrically stable* if the vector bundle $E_\mathbb{C}$ over $X_\mathbb{C}$ is stable. Not every stable vector bundle E over X is geometrically stable, but it is always geometrically polystable. Fix integers $r \geq 2$ and d. We denote by

$$\mathcal{M}(r,d) \supset \mathcal{M}(r,d)^s \longrightarrow M(r,d) \tag{1}$$

the moduli stack of vector bundles E over X of rank r and degree d, the open substack of geometrically stable E, and their coarse moduli scheme, respectively. Since geometrically stable E have only scalar automorphisms, $\mathcal{M}(r,d)^s$ is a gerbe with band $\mathbb{G}_\mathrm{m}$ over $M(r,d)$.

Let $\mathcal{L}(\det)$ denote the determinant of cohomology line bundle over $\mathcal{M}(r,d)$. Its fiber over the moduli point of a vector bundle E is by definition $\det \mathrm{H}^0(E) \otimes \det^{-1} \mathrm{H}^1(E)$.

All three moduli spaces or stacks in (1) come with a determinant map to the Picard variety $\underline{\mathrm{Pic}}^d(X)$. Given a real point ξ of $\underline{\mathrm{Pic}}^d(X)$, we denote by

$$\mathcal{M}(r,\xi) \supset \mathcal{M}(r,\xi)^s \longrightarrow M(r,\xi)$$

the corresponding fibers over ξ. So $M(r,\xi)$ is a smooth quasiprojective variety over $\mathbb{R}$, whose base change $M(r,\xi)_{\mathbb{C}}$ is the moduli space of stable vector over $X_{\mathbb{C}}$ of rank r and determinant $\xi_{\mathbb{C}}$. By restriction, $\mathcal{M}(r,\xi)^s$ is a gerbe with band $\mathbb{G}_{\mathrm{m}}$ over $M(r,\xi)$.

Suppose for the moment that ξ is a real line bundle. Then we can define a line bundle $\mathcal{L}(\xi)$ over $\mathcal{M}(r,\xi)$ whose fiber over the moduli point of a vector bundle E is $\mathrm{Hom}(\xi, \det E)$. To state this more precisely, let S be a scheme over $\mathbb{R}$. Then the pullback of $\mathcal{L}(\xi)$ along the classifying morphism $S \longrightarrow \mathcal{M}(r,\xi)$ of a vector bundle $\mathcal{E}$ over $X \times S$ is by definition the line bundle $\mathrm{pr}_{2,*}(\mathrm{pr}_1^*\xi^{-1} \otimes \det \mathcal{E})$ over S. This defines a line bundle $\mathcal{L}(\xi)$ over $\mathcal{M}(r,\xi)$.

Now suppose that ξ is quaternionic. Then the same recipe defines a line bundle over $\mathcal{M}(r,\xi)_{\mathbb{C}}$ endowed with a quaternionic structure. We denote this pair again by $\mathcal{L}(\xi)$.

In both cases, $\mathcal{L}(\xi)$ gives us a line bundle $\mathcal{L}(\xi)_{\mathbb{C}}$ over $\mathcal{M}(r,\xi)_{\mathbb{C}}$. If we trivialize the fiber of $\xi_{\mathbb{C}}$ over one closed point $x_0 \in X_{\mathbb{C}}$, we can identify $\mathcal{L}(\xi)_{\mathbb{C}}$ with the line bundle whose fiber at the moduli point of a vector bundle $E_{\mathbb{C}}$ over $X_{\mathbb{C}}$ is the fiber of $\det E_{\mathbb{C}}$ over x_0.

Proposition 2.1. *The Picard group* $\mathrm{Pic}(\mathcal{M}(r,\xi))$ *is generated*

i) *by* $\mathcal{L}(\det)$ *and* $\mathcal{L}(\xi)$*, if* ξ *is a real line bundle.*
ii) *by* $\mathcal{L}(\det)$ *and* $\mathcal{L}(\xi)^{\otimes 2}$*, if* ξ *is quaternionic.*

The restrictions of these line bundles also generate $\mathrm{Pic}(\mathcal{M}(r,\xi)^s)$*.*

Proof. Let $\widetilde{\mathcal{M}}(r,\xi_{\mathbb{C}})$ denote the moduli stack of vector bundles E of rank r over $X_{\mathbb{C}}$ together with an isomorphism $\xi_{\mathbb{C}} \cong \det E$. The forgetful map

$$\pi : \widetilde{\mathcal{M}}(r,\xi_{\mathbb{C}}) \longrightarrow \mathcal{M}(r,\xi)_{\mathbb{C}}$$

is the $\mathbb{G}_{\mathrm{m}}$-torsor given by the line bundle $\mathcal{L}(\xi)_{\mathbb{C}}$. It is easy to check that the kernel of

$$\pi^* : \mathrm{Pic}(\mathcal{M}(r,\xi)_{\mathbb{C}}) \longrightarrow \mathrm{Pic}(\widetilde{\mathcal{M}}(r,\xi_{\mathbb{C}}))$$

is generated by $\mathcal{L}(\xi)_{\mathbb{C}}$; cf. the proof of [BL, Lemma 7.8]. The Picard group of $\widetilde{\mathcal{M}}(r,\xi_{\mathbb{C}})$ is generated by $\pi^*(\mathcal{L}(\det)_{\mathbb{C}})$, according to [BL, Remark 7.11 and Proposition 9.2].

This shows that $\mathrm{Pic}(\mathcal{M}(r,\xi)_{\mathbb{C}})$ is generated by $\mathcal{L}(\det)_{\mathbb{C}}$ and $\mathcal{L}(\xi)_{\mathbb{C}}$. We have just seen that all these line bundles admit a real or quaternionic structure. This real or quaternionic structure is unique, since $\Gamma(\mathcal{M}(r,\xi)_{\mathbb{C}}, \mathcal{O}^*) = \mathbb{C}^*$. It follows that $\mathrm{Pic}(\mathcal{M}(r,\xi))$ is the subgroup of line bundles in $\mathrm{Pic}(\mathcal{M}(r,\xi)_{\mathbb{C}})$ which are real, not quaternionic. Hence $\mathrm{Pic}(\mathcal{M}(r,\xi))$ is generated by the line bundles as claimed.

As $\mathcal{M}(r,\xi)$ is smooth, the restriction map $\mathrm{Pic}(\mathcal{M}(r,\xi)) \longrightarrow \mathrm{Pic}(\mathcal{M}(r,\xi)^s)$ is surjective; cf. for example [BHo2, Lemma 7.3]. So these line bundles also generate $\mathrm{Pic}(\mathcal{M}(r,\xi)^s)$. □

Now let $\mathcal{M} \longrightarrow M$ be a gerbe with band $\mathbb{G}_{\mathrm{m}}$ over an irreducible Noetherian scheme M. As a basic example, we have the gerbe $\mathcal{M}(r,d)^s \longrightarrow M(r,d)$ in mind.

Definition 2.2. Let $\mathcal{L}$ be a line bundle over $\mathcal{M}$. Then the automorphism groups $\mathbb{G}_{\mathrm{m}}$ in $\mathcal{M}$ act on the fibers of $\mathcal{L}$. These $\mathbb{G}_{\mathrm{m}}$ act by the same power $w \in \mathbb{Z}$ on every fiber of $\mathcal{L}$, since $\mathcal{M}$ is connected. The integer w is called the *weight* of $\mathcal{L}$.

The weight of a quaternionic line bundle $\mathcal{L}$ is by definition the weight of the associated complex line bundle $\mathcal{L}_{\mathbb{C}}$. For example, the real or quaternionic line bundle $\mathcal{L}(\xi)$ over $\mathcal{M}(r,\xi)^s$ has weight r. The real line bundle $\mathcal{L}(\det)$ over $\mathcal{M}(r,d)^s$ has weight

$$\chi := r(1 - g_X) + d$$

according to Riemann-Roch. Consider the integers

$$\chi' := \chi/\gcd(r,\chi) \qquad \text{and} \qquad r' := r/\gcd(r,\chi).$$

The real or quaternionic line bundle

$$\mathcal{L}(\Theta) := \mathcal{L}(\det)^{\otimes -r'} \otimes \mathcal{L}(\xi)^{\otimes \chi'}$$

over $\mathcal{M}(r,\xi)^s$ has weight 0. Hence it descends to a real or quaternionic line bundle over $M(r,\xi)$, which we again denote by $\mathcal{L}(\Theta)$. The line bundle $\mathcal{L}(\Theta)_{\mathbb{C}}$ is ample on $M(r,\xi)_{\mathbb{C}}$, and it generates the Picard group $\mathrm{Pic}(M(r,\xi)_{\mathbb{C}})$ according to [DN, Théorèmes A & B].

Proposition 2.3. *The Picard group* $\mathrm{Pic}(M(r,\xi))$ *is generated*

i) *by* $\mathcal{L}(\Theta)$, *if* ξ *is a real line bundle or* χ' *is even.*
ii) *by* $\mathcal{L}(\Theta)^{\otimes 2}$, *if* ξ *is quaternionic and* χ' *is odd.*

Proof. The line bundles over $M(r,\xi)$ are the line bundles of weight 0 over $\mathcal{M}(r,\xi)^s$. According to Proposition 2.1, these are of the form $\mathcal{L}(\det)^{\otimes a} \otimes \mathcal{L}(\xi)^{\otimes b}$ with $a\chi + br = 0$, where moreover b has to be even if ξ is quaternionic. □

3. The Brauer group

The Brauer group $\mathrm{Br}(S)$ of a Noetherian scheme S is by definition the abelian group of Azumaya algebras over S up to Morita equivalence. It is a torsion group, and it embeds canonically into the étale cohomology group $\mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_{\mathrm{m}})$.

If S is smooth and quasiprojective over a field, then $\mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_{\mathrm{m}})$ is also a torsion group [Gr, Proposition 1.4], and the embedding of $\mathrm{Br}(S)$ into $\mathrm{H}^2_{\text{ét}}(S, \mathbb{G}_{\mathrm{m}})$ is an isomorphism [dJ].

Our aim is to compute the Brauer group of the real moduli space $M(r,\xi)$. Let

$$\beta \in \mathrm{H}^2_{\text{ét}}(M(r,\xi), \mathbb{G}_{\mathrm{m}}) = \mathrm{Br}(M(r,\xi)) \tag{2}$$

denote the class given by the gerbe $\mathcal{M}(r,\xi)^s \longrightarrow M(r,\xi)$ with band $\mathbb{G}_{\mathrm{m}}$. Since a section of this gerbe would yield a Poincaré bundle over $M(r,\xi) \times X$, we can view the class β as the obstruction against the existence of such a Poincaré bundle.

Remark 3.1. Choose an effective divisor $D \subset X$ defined over $\mathbb{R}$, for example a closed point in X. The Brauer class β over $M(r,\xi)$ can also be described by the Azumaya algebra with fibers $\mathrm{End}\,\mathrm{H}^0(D, E|_D)$, or by the projective bundle with fibers $\mathbb{P}\,\mathrm{H}^0(D, E|_D)$.

We first compute the exponent of β, i.e., the order of β as an element in the torsion group $\mathrm{Br}(M(r,\xi))$. This will in particular reprove results of [BHu, Section 5].

Proposition 3.2. *Let ξ be a real point of the Picard variety $\underline{\mathrm{Pic}}^d(X)$.*

i) *If ξ is a real line bundle, then $\beta \in \mathrm{Br}(M(r,\xi))$ has exponent* $\gcd(r,\chi)$.
ii) *If ξ is quaternionic, then $\beta \in \mathrm{Br}(M(r,\xi))$ has exponent* $\gcd(2r,\chi)$.

Proof. An integer $n \in \mathbb{Z}$ annihilates the class $\beta \in \mathrm{H}^2_{\text{ét}}(M(r,\xi),\mathbb{G}_{\mathrm{m}})$ of the gerbe $\mathcal{M}(r,\xi)^s$ if and only if there is a line bundle $\mathcal{L}$ over $\mathcal{M}(r,\xi)^s$ which has weight n; see for example [Ho, Lemma 4.9]. Hence the claim follows from Proposition 2.1. □

We denote by $\mathbb{Z}\cdot\beta \subseteq \mathrm{Br}(M(r,\xi))$ the subgroup generated by the class β in (2). Let

$$f : M(r,\xi) \longrightarrow \mathrm{Spec}(\mathbb{R}) \tag{3}$$

be the structure morphism. Recall that $\mathrm{Br}(\mathbb{R}) \cong \mathbb{Z}/2$, the nontrivial element being the class $[\mathbb{H}] \in \mathrm{Br}(\mathbb{R})$ of the quaternion algebra $\mathbb{H} = \mathbb{R}\oplus\mathbb{R}\cdot i\oplus\mathbb{R}\cdot j\oplus\mathbb{R}\cdot k$.

Theorem 3.3. *Let ξ be a real point of $\underline{\mathrm{Pic}}^d(X)$, with d odd if $g_X = r = 2$. We have*

$$\mathrm{Br}(M(r,\xi)) = \begin{cases} \mathbb{Z}\cdot\beta \oplus f^*(\mathrm{Br}(\mathbb{R})) \cong \mathbb{Z}/\gcd(r,\chi)\oplus\mathbb{Z}/2 & \textit{if } \xi \textit{ is a real line bundle,} \\ \mathbb{Z}\cdot\beta \cong \mathbb{Z}/\gcd(2r,\chi) & \textit{if } \xi \textit{ is quaternionic.} \end{cases}$$

Proof. The structure morphism f in (3) yields a Leray spectral sequence

$$E_2^{p,q} = \mathrm{H}^p_{\text{ét}}(\mathbb{R}, \mathrm{R}^q f_*\mathbb{G}_{\mathrm{m}}) \Rightarrow \mathrm{H}^{p+q}_{\text{ét}}(M(r,\xi),\mathbb{G}_{\mathrm{m}}). \tag{4}$$

We have $\mathrm{R}^1 f_*\mathbb{G}_{\mathrm{m}} = \mathrm{Pic}(M(r,\xi)_{\mathbb{C}}) \cong \mathbb{Z}$. The action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2$ on it is trivial, for example because it preserves ampleness. From this we deduce

$$E_2^{1,1} = \mathrm{H}^1_{\text{ét}}(\mathbb{R},\mathbb{Z}) = \mathrm{Hom}(\mathbb{Z}/2,\mathbb{Z}) = 0.$$

Hence the spectral sequence (4) provides in particular an exact sequence

$$\mathrm{H}^1_{\text{ét}}(M(r,\xi),\mathbb{G}_{\mathrm{m}}) \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0} \longrightarrow \mathrm{H}^2_{\text{ét}}(M(r,\xi),\mathbb{G}_{\mathrm{m}}) \longrightarrow E_2^{0,2}.$$

Using $f_*\mathbb{G}_{\mathrm{m}} = \mathbb{G}_{\mathrm{m}}$ and $\mathrm{R}^2 f_*\mathbb{G}_{\mathrm{m}} = \mathrm{Br}(M(r,\xi)_{\mathbb{C}})$, we thus obtain an exact sequence

$$\mathrm{Pic}(M(r,\xi)) \xrightarrow{g^1} \mathrm{Pic}(M(r,\xi)_{\mathbb{C}}) \longrightarrow \mathrm{Br}(\mathbb{R}) \xrightarrow{f^*} \mathrm{Br}(M(r,\xi)) \xrightarrow{g^2} \mathrm{Br}(M(r,\xi)_{\mathbb{C}})$$

where g^1 and g^2 are pullback maps along the projection $g : M(r,\xi)_{\mathbb{C}} \longrightarrow M(r,\xi)$. Note that g^2 is surjective, since $g^2(\beta) = \beta_{\mathbb{C}}$ generates $\mathrm{Br}(M(r,\xi)_{\mathbb{C}})$ by [BBGN].

Suppose that ξ is a real line bundle. Then g^1 is surjective due to Proposition 2.3, so f^* is injective. Since β has the same exponent as its image $\beta_{\mathbb{C}}$ by Proposition 3.2, it follows that $\mathrm{Br}(M(r,\xi))$ is the direct sum of its subgroups $\mathbb{Z}\cdot\beta$ and $f^*(\mathrm{Br}(\mathbb{R}))$, as required.

Now suppose that ξ is quaternionic and that $\chi' = \chi/\gcd(r,\chi)$ is even. Then f^* is injective as before, but the exponent $\gcd(2r,\chi)$ of β is twice the exponent $\gcd(r,\chi)$ of its image $\beta_{\mathbb{C}}$. Hence $\gcd(r,\chi)\cdot\beta = f^*([\mathbb{H}])$, and the class β generates $\mathrm{Br}(M(r,\xi))$.

Finally, suppose that ξ is quaternionic and that χ' is odd. Then the cokernel of g^1 has two elements according to Proposition 2.3, so f^* is the zero map, and g^2 is an isomorphism. In particular, the class β again generates $\mathrm{Br}(M(r,\xi))$. □

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