Stability Concepts in Algebraic Geometry

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Abstract

This thesis studies stability concepts in Algebraic Geometry. The notion of stability comes from Geometric Invariant Theory, introduced by D. Mumford, in order to construct moduli spaces. More recently, the space of stability conditions on a derived category was introduced by T. Bridgeland. In this thesis, we outline preliminaries such as complex manifolds, divisors and line bundles, sheaf theory and category theory. We then study the stability of vector bundles on curves of small genus and give an overview of Bridgeland stability conditions. We conclude by examining stability conditions on the category of coherent systems and the category of holomorphic triples in this framework.

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Introduction

Algebraic geometry has developed tremendously over the last century. During the 19th century, the subject was practiced on a relatively concrete level. However, towards the end of that century, much advancement was made in this discipline, especially by Italian geometers. Then in the 1950's, Serre introduced sheaf theory. The foundations of Algebraic Geometry have been brought to modern standards by work of E. Noether, A. Weil, O. Zariski and others. In the 1960s, A. Grothendieck introduced the currently used language of schemes and many fundamental ideas, which are now established as standard concepts.

Moduli spaces, introduced by B. Riemann, form a major tool of modern algebraic geometry. Stability concepts of geometric objects play a crucial role in the construction and study of moduli spaces. Stability concepts emerged from Geometric Invariant Theory, which is used to construct moduli spaces. These concepts were originally introduced by Mumford, Maruyama and others. More recently, inspired by ideas from theoretical physics namely string theory, T. Bridgeland introduced a generalisation of the notion of stability to derived categories. The resulting moduli space of Bridgeland's stability conditions is an interesting and new invariant. As this construction is relatively new, only a few examples are studied so far, see for example the works of T. Bridgeland, D. Arcara, E. Macri, S. Okada and others.

This thesis focuses on a very small, yet relevant, part of this vast and rapily evolving subject area. Firstly, it aims to provide an overview of some existing stability concepts in different contexts of algebraic geometry. To this end, in the first three chapters we need to introduce notions such as complex manifolds, sheaves, categories and vector bundles.

The paper is organised as follows. In Chapter 1, we outline the necessary background and preliminaries required for subsequent chapters. We begin with a brief review of complex manifolds and then go on to describe divisors and line bundles on a complex manifold. We also give a brief exposition of sheaf theory, followed by category theory. This is the language we use for the rest of the thesis.

Chapter 2 is devoted to the stability of vector bundles on curves. The first section outlines basic definitions and theorems. We will then study vector bundles on $\mathbb{P}^1$. From here, we go on to study Atiyah's classification of indecomposable rank two vector bundles over an elliptic curve. The final section introduces the notion of stability of vector bundles.

We give an overview of Bridgeland stability concepts following his exposition in [Br01] in Chapter 3. This concludes the foundational material needed for the following two chapters. In Chapter 4 we study coherent systems on a Riemann surface in an attempt to place the stability of coherent systems into the framework of Bridgeland stability conditions. We discover that, in fact, the category of coherent systems does not form an abelian category and so we cannot go any further. This leads us to study holomorphic triples in the final chapter. We check again if the stability of triples can be placed into the framework of Bridgeland stability conditions. Here, we make some interesting conclusions, although many open questions still remain. Throughout this thesis we work over $\mathbb{C}$, the field the complex numbers.

Contents

Chapter 1

Preliminaries

1.1 Complex manifolds

Throughout this thesis we work over complex manifolds. Complex manifolds are topological spaces that locally look like $\mathbb{C}^n$. They are close relatives of differentiable manifolds, but very different in many aspects. In this section, knowledge of differentiable manifolds is assumed. Let us now look at the precise definition of a complex manifold.

Definition 1.1.1. An *n-dimensional complex manifold*, X, is a differentiable manifold admitting an open cover $\{U_i\}$ and coordinate maps $\varphi_i : U_i \to \mathbb{C}^n$ such that

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is holomorphic for all i, j.

Remark 1.1.2. We write $\varphi_i^{-1} : \varphi_i(U_i) \to U_i \subset X$ for the inverse of φ_i on it's image $\varphi_i(U_i)$.

Remark 1.1.3. Using the notation of Definition 1.1.1, denote by $V_i := \varphi_i(U_i)$, $V_{ij} := \varphi_i(U_i \cap U_j), V_{ji} := \varphi_j(U_i \cap U_j) \subset \mathbb{C}^n$. Define $\varphi_i \circ \varphi_j^{-1} =: \psi_{ij} : V_{ji} \to V_{ij}$. These are holomorphic and satisfy

$$\psi_{ij} \circ \psi_{ji} = \mathrm{id} \quad \text{on } V_{ij}$$

and

$$\psi_{ij} \circ \psi_{jk} \circ \psi_{ki} = \mathrm{id} \qquad \text{on } V_{ij} \cap V_{ik}.$$

We call these maps, the *change of coordinate maps* of X.

A function, f, on an open set $U \subset X$ is *holomorphic* if, for all i, $f \circ \varphi_i^{-1}$ is holomorphic on $\varphi_i(U \cap U_i) \subset \mathbb{C}^n$.

Definition 1.1.4. A complex manifold is called connected, compact etc. if the underlying differentiable (or topological) manifold has this property. A complex manifold of dimension one is called a *Riemann surface*.

Let us now look at some examples of complex manifolds.

Example 1.1.5. The simplest example of a one dimensional complex manifold is just $\mathbb{C}$ itself. The affine space is another example, which is just the algebraic name for the most basic complex manifold provided by the n-dimensional complex space $\mathbb{C}^n$. Any complex vector space of finite dimension is also a complex manifold.

Example 1.1.6. The complex projective space $\mathbb{P}^n := \mathbb{P}^n_{\mathbb{C}}$ is an example of a compact complex manifold. By definition, $\mathbb{P}^n$ is the set of equivalence classes of $(n+1)$-tuples $(a_0, \dots, a_n)$ of elements of $\mathbb{C}$, not all zero, under the equivalence relation given by $(a_0, \dots, a_n) \sim (\lambda a_0, \dots, \lambda a_n)$ for all $\lambda \in \mathbb{C}^*$. Another way of saying this is that $\mathbb{P}^n$ as a set is the quotient of the set $\mathbb{C}^{n+1} \backslash \{(0, \dots, 0)\}$ under the equivalence relation which identifies points lying on the same line through the origin.

An element of $\mathbb{P}^n$ (i.e. an equivalence class of $(n+1)$-tuples of elements of $\mathbb{C}$) is called a point and written as $(z_0 : z_1 : \dots : z_n)$. The *standard open covering* of $\mathbb{P}^n$ is given by $n+1$ open subsets

$$U_i := \{(z_0 : \dots : z_n) | z_i \neq 0\} \subset \mathbb{P}^n.$$

If $\mathbb{P}^n$ is endowed with the quotient topology via

$$\pi : \mathbb{C}^{n+1} \backslash \{0\} \to (\mathbb{C}^{n+1} \backslash \{0\}) / \mathbb{C}^* = \mathbb{P}^n,$$

then the U_i's are indeed open. The structure of a complex manifold on this set is given by the following: On the open subset $U_i \subset \mathbb{P}^n$, there are bijective maps

$$\varphi_i : U_i \to \mathbb{C}^n, (z_0 : \ldots : z_n) \mapsto \left(\frac{z_0}{z_i}, \cdots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \cdots, \frac{z_n}{z_i} \right).$$

On $\{w_i \neq 0\} = \varphi_j(U_i \cap U_j) \subset \mathbb{C}^n$, the change of coordinate maps $\psi_{ij} := \varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$, are holomorphic. They are given by

$$\psi_{ij}(w_1, \ldots, w_n) = \left(\frac{w_1}{w_{i+1}}, \ldots, \frac{w_i}{w_{i+1}}, \frac{w_{i+2}}{w_{i+1}}, \ldots, \frac{w_j}{w_{i+1}}, \frac{1}{w_{i+1}}, \frac{w_{j+1}}{w_{i+1}}, \ldots, \frac{w_n}{w_{i+1}} \right),$$

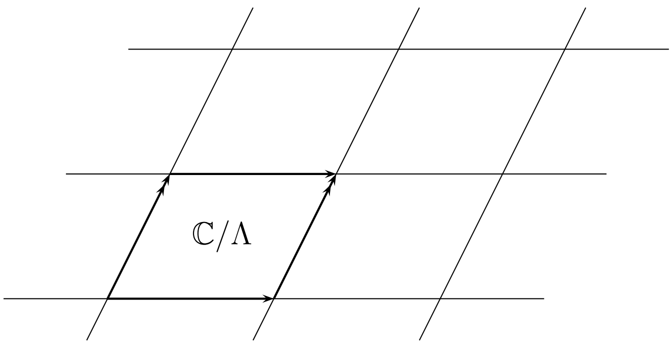
if $i < j$ and

$$\psi_{ij}(w_1, \ldots, w_n) = \left(\frac{w_1}{w_i}, \ldots, \frac{w_j}{w_i}, \frac{1}{w_i}, \frac{w_{j+1}}{w_i}, \ldots, \frac{w_{i-1}}{w_i}, \frac{w_{i+1}}{w_i}, \ldots \frac{w_n}{w_i} \right),$$

if $i > j$. Thus $\mathbb{P}^n$ has the structure of a complex manifold.

Example 1.1.7. Let X be the quotient $\mathbb{C}^n/\mathbb{Z}^{2n}$, where $\mathbb{Z}^{2n} \subset \mathbb{R}^{2n} = \mathbb{C}^n$ is the natural inclusion. Then X has the structure of a complex manifold induced by the projection map $\pi : \mathbb{C}^n \to \mathbb{C}^n/\mathbb{Z}^{2n}$. This complex manifold is called a *complex torus.*

Let us consider the one-dimensional case, i.e. $X = \mathbb{C}/\Lambda$, where $\Lambda = \mathbb{Z}^2 = \{n_1 w_1 + n_2 w_2 | n_i \in \mathbb{Z}, w_i \in \mathbb{C}\}$ is a rank two lattice in $\mathbb{C}$. Since $\mathbb{R}/\mathbb{Z}$ is diffeomorphic to S^1 via the exponential map $r \mapsto \exp(2\pi i r)$, where $r \in \mathbb{R}, \mathbb{C}/\Lambda$ is diffeomorphic to $S^1 \times S^1$. This one dimensional complex torus is also called an *elliptic curve.*



Definition 1.1.8. Let X be an n-dimensional complex manifold and let $Y \subset X$ be a differentiable submanifold of real dimension $2k$. Then Y is a *complex submanifold* of X if there exists an open cover $\{U_i\}$ of X and coordinate maps $\varphi_i : U_i \to \mathbb{C}^n$ of X such that $\varphi_i(U_i \cap Y) \cong \varphi_i(U_i) \cap \mathbb{C}^k$.

Here $\mathbb{C}^k$ is embedded into $\mathbb{C}^n$ via $(z_1, \ldots, z_k) \mapsto (z_1, \ldots, z_k, 0, \ldots, 0)$.

Definition 1.1.9. A complex manifold X is *projective* if X is isomorphic to a closed complex submanifold of some projective space $\mathbb{P}^N$.

Each compact Riemann surface can be embedded holomorphically into some projective space $\mathbb{P}^N$. A compact Riemann surface S together with an embedding $i : S \hookrightarrow \mathbb{P}^N$ is known as an *algebraic curve*. In this paper, we will often refer to a compact Riemann surface as a *curve* without necessarily specifying the embedding in $\mathbb{P}^N$.

1.2 Divisors

There is an important notion in algebraic geometry which is that of a divisor. In the case of a curve it has a simple description as follows:

Definition 1.2.1. Let C be a smooth projective curve. A *divisor* on C is a formal linear combination $D = a_1P_1 + \cdots + a_mP_m$ of points $P_i \in C$ with integer coefficients a_i. Divisors can be added or subtracted and hence form a group denoted $\mathrm{Div}(C)$.

Remark 1.2.2. This definition can also be extended to higher dimensional manifolds. On a complex manifold, X, of dimension n, a divisor is a formal linear combination of closed subvarieties of dimension $n-1$. A closed subvariety Y of dimension $n-1$ in Y is a subset of X which can locally be given as the zero locus of a single holomorphic function.

Definition 1.2.3. The *degree* of a divisor $D = a_1P_1 + \cdots + a_mP_m$ on a curve, C, is defined to be $\deg D = \sum_{i=1}^{m} a_i$ and gives us a group homomorphism $\deg : \mathrm{Div}(C) \to \mathbb{Z}$.

Let f be a holomorphic function on an open set $U \subset C$. Let $P \in U$ and let x be the local coordinate on U such that $x(P) = \lambda$ for some $\lambda \in \mathbb{C}$. We define the *order of f at P*, denoted $\mathrm{ord}_P f$ to be the largest $a \in \mathbb{Z}$ such that locally

$$f(x) = (x - \lambda)^a \cdot h(x) \quad \text{where } h \text{ is a holomorphic function,} \quad h(\lambda) \neq 0.$$

Note that for g, h any holomorphic functions

$$\mathrm{ord}_P gh = \mathrm{ord}_P g + \mathrm{ord}_P h.$$

Now let f be a meromorphic function on C not identically zero, i.e. f can be written locally as a ratio $\frac{g}{h}$, where g and h are holomorphic functions which do not have a common zero. We define

$$\mathrm{ord}_P f = \mathrm{ord}_P g - \mathrm{ord}_P h.$$

Given a global meromorphic function we can construct a divisor associated to it.

Definition 1.2.4. Let f be a meromorphic function on C. Then the divisor associated to f, called a *principal divisor* and denoted $\mathrm{div}(f)$ is

$$\mathrm{div}(f) = \sum_{P \in C} \mathrm{ord}_P f \cdot P.$$

Example 1.2.5. Consider $\mathbb{P}^1$ with homogeneous coordinates $(z_0 : z_1)$. Then any ratio $f = \frac{g}{h}$ where g and h are homogeneous polynomials of degree d defines a global meromorphic function. For example, if $f = \frac{z_0 z_1}{z_0^2 - z_1^2}$ then $\mathrm{div}(f) = P_0 + P_1 - P_2 - P_3$ where $P_0 = (1:0), P_1 = (0:1), P_2 = (1:1)$ and $P_3 = (1:-1)$. Note that $\deg(\mathrm{div}(f)) = 0$.

1.2.1 Line bundles and divisors

We will now see how to relate divisors to holomorphic line bundles. From now on X will denote a complex manifold, unless otherwise specified. We begin with the definition of a line bundle.

Definition 1.2.6. A *holomorphic line bundle* is a holomorphic map $p : L \to X$ of complex manifolds which satisfies the following conditions:

1. For any point $x \in X$, the preimage $L_x := p^{-1}(x)$ (called a *fibre*) has a structure of a one-dimensional $\mathbb{C}$-vector space.

2. The mapping p is locally trivial, i.e. for any point $x \in X$, there exists an open neighbourhood U_i containing x and a biholomorphic map $\varphi_i : p^{-1}(U_i) \to U_i \times \mathbb{C}$ such that the diagram

$$\begin{tikzcd} p^{-1}(U_i) \arrow[rr, "\varphi_i"] \arrow[dr, "p"'] & & U_i \times \mathbb{C} \arrow[dl, "\mathrm{pr}_1"] \\ & U_i & \end{tikzcd}$$

commutes (where pr_1 is projection to the first factor).

Moreover, φ_i takes the vector space L_x isomorphically onto $\{x\} \times \mathbb{C}$ for each $x \in U_i$; φ_i is called a *trivialisation* of L over U_i. Note that for any pair of trivialisations φ_i and φ_j the map

$$g_{ij} : U_i \cap U_j \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$$

given by

$$g_{ij}(x) = \varphi_i \circ (\varphi_j)^{-1}|_{\{x\}\times\mathbb{C}}, \text{ i.e. } \varphi_i(\varphi_j^{-1}(x, v)) = (x, g_{ij}(x)v)$$

is holomorphic; the maps g_{ij} are called *transition functions* for L relative to the trivialisations φ_i, φ_j. The transition functions of L necessarily satisfy the identities

$$g_{ij}(x) \cdot g_{ji}(x) = 1 \qquad \text{for all } x \in U_i \cap U_j \tag{1.1}$$

$$g_{ij}(x) \cdot g_{jk}(x) \cdot g_{ki}(x) = 1 \qquad \text{for all } x \in U_i \cap U_j \cap U_k. \tag{1.2}$$

Conversely, given an open cover $\{U_i\}$ of X and transition functions $g_{ij} : U_i \cap U_j \to \mathrm{GL}(n, \mathbb{C})$, for all i, j, satisfying conditions (1.1) and (1.2) outlined above, then we can define a line bundle, L, with transition functions g_{ij} using the glueing construction as follows: We glue $U_i \times \mathbb{C}$ together by taking the union over all i of $U_i \times \mathbb{C}$ to get $L := \bigsqcup (U_i \times \mathbb{C})/\sim$, where $(x, v) \sim (x, g_{ij}(x)(v))$, for all $x \in U_i \cap U_j, v \in \mathbb{C}$.

Example 1.2.7. The trivial line bundle on X, i.e. $\mathrm{pr}_1 : X \times \mathbb{C} \to X$ will be denoted by $\mathcal{O}_X$, (or simply $\mathcal{O}$ if it is clear which X we are referring to).

Remark 1.2.8. If we consider line bundles on $\mathbb{P}^1$, with coordinates $z = (z_0 : z_1)$ and with standard open cover $\{U_0, U_1\}$ as outlined in Example 1.1.6, it is enough to give the transition function $g_{01} : U_0 \cap U_1 \to \mathrm{GL}(1, \mathbb{C})$. From this we can define $g_{10} : U_1 \cap U_0 \to \mathrm{GL}(1, \mathbb{C})$ as follows: $g_{10}(z) := g_{01}(z)^{-1}$ and $g_{00}(z) = 1$ and $g_{11}(z) = 1$. The transition functions then clearly satisfy the necessary conditions:

$$g_{01}(z) \cdot g_{10}(z) = 1$$

$$g_{ij}(z) \circ g_{jk}(z) \circ g_{ki}(z) = 1 \quad \text{for all } i, j, k.$$

Example 1.2.9. Consider the line bundle on $\mathbb{P}^1$ given by the transition function $g_{01} = \frac{z_1^n}{z_0^n}$. This line bundle is normally denoted $\mathcal{O}(n)$. Using this convention we see that the trivial line bundle $\mathcal{O}$ can also be written as $\mathcal{O}(0)$.

Definition 1.2.10. Given two line bundles L_1 and L_2 on X (with open cover $\{U_i\}$), with transition functions g_{ij} and h_{ij} respectively, then we can define $L_1 \otimes L_2$, the *tensor product* of L_1 and L_2, to be the line bundle with transition functions

$$f_{ij}(x) := g_{ij}(x) \cdot h_{ij}(x) \in \mathrm{GL}(1, \mathbb{C}) \quad \forall x \in U_i \cap U_j$$

Example 1.2.11. Consider the line bundles $\mathcal{O}(n)$ and $\mathcal{O}(m)$ on $\mathbb{P}^1$. Let $g_{01} = \frac{z_1^n}{z_0^n}$ and $h_{01} = \frac{z_1^m}{z_0^m}$ be transition functions for $\mathcal{O}(n)$ and $\mathcal{O}(m)$, respectively. So we get the line bundle $\mathcal{O}(n) \otimes \mathcal{O}(m)$ given by transition functions $f_{01} = g_{01} \cdot h_{01}$, i.e $f_{01} = \frac{z_1^{n+m}}{z_0^{n+m}}$ and so we can clearly see that

$$\mathcal{O}(n) \otimes \mathcal{O}(m) \cong \mathcal{O}(n+m).$$

Definition 1.2.12. If L is a line bundle on X with transition functions g_{ij}, and $\{U_i\}$ is an open cover on X, then the *dual bundle*, L^*, of L is given by transition functions

$$h_{ij}(x) := (g_{ij}(x))^{-1} \quad \forall x \in U_i \cap U_j.$$

Example 1.2.13. Consider the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$ (as described in Example 1.2.9) with transition functions $g_{01} = \frac{z_1^n}{z_0^n}$. Then $\mathcal{O}(n)^*$, the dual of this, is the line bundle given by transition functions

$$h_{01}(x) = \frac{z_0^n}{z_1^n} = \frac{z_1^{-n}}{z_0^{-n}}$$

So we can see that $\mathcal{O}(n)^* \cong \mathcal{O}(-n)$.

Remark 1.2.14. The set $\mathrm{Pic}(X)$ of isomorphism classes of line bundles over a complex manifold X is a group, called the *Picard group*, with respect to the operation of tensoring. The trivial line bundle is the neutral element. For any line bundle L its *dual bundle* L^* is the inverse, i.e. $L \otimes L^* \cong \mathcal{O}$.

Definition 1.2.15. Let $p : L \to X$ be a line bundle with transition functions g_{ij}. A *holomorphic section* s of the line bundle $p : L \to X$ is a holomorphic map $s : X \to L$ such that $p \circ s = \mathrm{id}$. This means that we have an open cover $\{U_i\}$ of X and a collection of holomorphic functions $s_i : U_i \to \mathbb{C}$ such that

$$s_i(x) = g_{ij}(x)s_j(x) \quad \forall x \in U_i \cap U_j.$$

Note that it may turn out that a line bundle does not have any holomorphic sections. We can consider meromorphic sections of L. A collection of local meromorphic functions $\{s_i\}$ on $\{U_i\}$ such that $s_i(x) = g_{ij}(x)s_j(x)$ for all $x \in U_i \cap U_j$ is called a *meromorphic section* of L.

We are now ready to give the correspondence between divisors and line bundles. Let $D = \sum n_i P_i$ be a divisor on a smooth projective curve, C. Let $\{U_i\}$ be an open cover of C. If z is a coordinate on U_i with $z(P_j) = \lambda_j$, then

$$f_i(z) = \prod_{P_j \in U_i} (z - \lambda_j)^{n_j}$$

defines a meromorphic function on U_i with $\mathrm{div}(f_i) = D|_{U_i}$, the part of D which is inside U_i. We say that D is locally defined by f_i. Then the functions

$$g_{ij} = \frac{f_i}{f_j}$$

are holomorphic and nonzero on $U_i \cap U_j$ because on $U_i \cap U_j$, f_i and f_j have the same poles and zeros of the same order. On $U_i \cap U_j \cap U_k$, we have

$$g_{ij} \cdot g_{jk} \cdot g_{ki} = \frac{f_i}{f_j} \cdot \frac{f_j}{f_k} \cdot \frac{f_k}{f_i} = 1.$$

The line bundle given by the transition functions $\{g_{ij} = f_i/f_j\}$ is called the *associated line bundle* of D, denoted $\mathcal{O}(D)$. We check that it is well-defined: if $\{f_i'\}$ are alternate local data for D, then $h_i = f_i/f_i'$ is a nonzero holomorphic function on U_i and

$$g_{ij}' = \frac{f_i'}{f_j'} = g_{ij} \cdot \frac{h_j}{h_i}$$

for each i, j.

So now we can define a map

$$\mathscr{L} : \mathrm{Div}(C) \to \mathrm{Pic}(C)$$

given by

$$D \mapsto \mathcal{O}(D)$$

The correspondence $\mathscr{L}$ has these immediate properties: First, if D and D' are two divisors given by local data $\{f_i\}$ and$\{f_i'\}$, respectively, then $D + D'$ is given by $\{f_i \cdot f_i'\}$. Now $\mathcal{O}(D)$ is given by transition functions $g_{ij} = f_i/f_j$ and $\mathcal{O}(D')$ is given by transition functions $g_{ij}' = f_i'/f_j'$. So $\mathcal{O}(D+D')$ is given by transition functions $g_{ij} \cdot g_{ij}'$. Hence, by definition of the tensor product of line bundles (Definition 1.2.10) it follows that

$$\mathcal{O}(D + D') \cong \mathcal{O}(D) \otimes \mathcal{O}(D')$$

so the map

$$\mathscr{L} : \mathrm{Div}(C) \to \mathrm{Pic}(C)$$

is a homomorphism.

Let us now show that $\mathscr{L}$ is surjective. Since C can be embedded in some projective space $\mathbb{P}^N$, given a line bundle L on C, there exists a meromorphic section s of L (this follows from [H] Theorem II.5.17). Consider a local representation s_i of s. Then given any point $P \in C$ we can define the *order of s at P* as

$$\operatorname{ord}_P(s) = \operatorname{ord}_P(s_i)$$

for any i such that $P \in U_i$. Since $\frac{s_i(x)}{s_j(x)} = g_{ij}(x) \in \mathbb{C}^* \ \forall x \in U_i \cap U_j$, this does not depend on the choice of i and it follows that $\operatorname{ord}_P(s_i) = \operatorname{ord}_P(s_j)$ if $P \in U_i \cap U_j$. Hence $\operatorname{ord}_P(s)$ is well-defined. We take the divisor, $\operatorname{div}(s)$ of s to be

$$\operatorname{div}(s) = \sum_{P \in C} \operatorname{ord}_P s \cdot P.$$

With this convention s is holomorphic if and only if $\operatorname{div}(s)$ is effective, i.e. $\operatorname{ord}_P(s) \geq 0$ for all $P \in C$. If we were to take the line bundle associated to the divisor $\operatorname{div}(s)$ we would recover (up to isomorphism) our original line bundle, L. In light of this, it is common to denote a line bundle by $\mathcal{O}(D)$.

The third property of $\mathscr{L}$ is that it's kernel is exactly the principal divisors. This is given by the following lemma:

Lemma 1.2.16. *Let C be a smooth projective curve and let* $\operatorname{Div}(C)$ *denote the group of divisors and* $\operatorname{Pic}(C)$ *denote the Picard group on C. Then $\mathcal{O}(D)$ is trivial if and only if $D = \operatorname{div}(f)$ for some meromorphic function on C.*

Proof. ([GH] Chapter 1, Section 1, Pg. 134) If $D = \operatorname{div}(f)$ for some meromorphic function f on C, we may take as local data for D over any cover $\{U_i\}$ the functions $f_i = f|_{U_i}$; then $f_i/f_j = 1$ and so $\mathcal{O}(D)$ is trivial. Conversely, if D is given by local data $\{f_i\}$ and the line bundle $\mathcal{O}(D)$ is trivial, then there exists nonzero holomorphic functions h_i on U_i, corresponding to a constant section of the trivial line bundle, such that

$$\frac{f_i}{f_j} = g_{ij} = \frac{h_i}{h_j};$$

the function $f = f_i \cdot h_i^{-1} = f_j \cdot h_j^{-1}$ is then a global meromorphic function on C with divisor D. $\square$

Definition 1.2.17. We say that two divisors D, D' on C are *linearly equivalent* and write $D \sim D'$ if $D = D' + \operatorname{div}(f)$ for some f, a meromorphic function on C or equivalently if $\mathcal{O}(D) \cong \mathcal{O}(D')$.

It follows from Lemma 1.2.16 that if D is a principal divisor, $\mathcal{O}(D) \cong \mathcal{O}(0)$, where $\mathcal{O}(0)$ is the trivial line bundle, then $\deg(D) = 0$. We have already seen (Definition 1.2.3) that

$$\deg : \operatorname{Div}(C) \to \mathbb{Z}$$

is a homomorphism of groups. It is zero on principal divisors, and so it factors through $\operatorname{Pic}(C)$ to give a homomorphism

$$\deg : \operatorname{Pic}(C) \to \mathbb{Z}.$$

We can now give the following definition.

Definition 1.2.18. The *degree* of a line bundle $\mathcal{O}(D)$, denoted $\deg(\mathcal{O}(D))$ is the degree of the divisor D, i.e. $\deg(\mathcal{O}(D)) = \deg(D)$.

Remark 1.2.19. The process of constructing the map $\operatorname{Div}(C) \to \operatorname{Pic}(C)$ on a smooth projective curve can also be carried out for divisors on higher dimensional manifolds, though we do not go into the details here.

1.3 Sheaves

Sheaves are a powerful tool used to deduce global properties from local ones. Jean Leray invented the theory of sheaves on topological spaces. However, it was Serre who introduced sheaf theory into algebraic geometry. The theory of sheaves play a large role in this thesis and this section seeks to outline the main properties that we will need in later chapters.

Throughout this section we will work over complex manifolds. However, all the definitions could also be made over a topological space, though we do not need this level of generality for this thesis. When we refer to 'rings' in this section, we mean commutative rings with identity, unless otherwise specified.

Definition 1.3.1. Let X be a complex manifold. A *presheaf*, $\mathcal{F}$, of abelian groups consists of the following data:

- for every open subset U of X an abelian group $\mathcal{F}(U)$ and
- for every inclusion $V \subseteq U$ of open sets in X a group homomorphism

$$\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$$

such that

(a) $\mathcal{F}(\emptyset) = 0$;
(b) $\rho_{U,U} = \mathrm{id}_{\mathcal{F}(U)}$, and
(c) for open sets $W \subseteq V \subseteq U$ of X

$$\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}.$$

The homomorphism $\rho_{U,V}$ is called a *restriction map* and sometimes we write $s|_V$ instead of $\rho_{U,V}(s)$, if $s \in \mathcal{F}(U)$.

Furthermore, a presheaf $\mathcal{F}$ of abelian groups is called a *sheaf* of abelian groups if it satisfies the following glueing property: if $U \subset X$ is an open set, $\{U_i\}$ an open cover of U and $s_i \in \mathcal{F}(U_i)$ for all i such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, then there is a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

We also write $\mathcal{F}(U)$ as $\Gamma(U, \mathcal{F})$. An element of $\Gamma(U, \mathcal{F})$ is said to be a *section* of $\mathcal{F}$ over U.

Remark 1.3.2. We can also define a sheaf of vector spaces on X, where for every open subset $U \subseteq X$, the $\mathcal{F}(U)$ are vector spaces and the restriction maps are linear maps. Similarly we can define a sheaf of rings (i.e. commutative rings with identity) on X, where for every open subset $U \subseteq X$, the $\mathcal{F}(U)$ are rings and the restriction maps are ring homomorphisms.

Example 1.3.3. Let U be an open set in a complex manifold X, and let $C_X(U)$ be the totality of real-valued continuous functions on U. Define, for $x \in U$,

$$(f+g)(x) = f(x) + g(x) \text{ and } (fg)(x) = f(x)g(x), \quad f, g \in C_X(U).$$

Then $C_X(U)$ becomes a commutative ring with the identity being the constant function 1. For open sets $V \subseteq U$, the restriction map $\rho_{U,V}$ is the restriction of the domain of a function, i.e. $f \mapsto f\,|_V$. Then C_X is a sheaf of rings over the complex manifold X.

Intuitively speaking, any set of "function-like" objects form a presheaf; it is a sheaf if the conditions imposed on the "functions" are local.

Example 1.3.4. Let $X = \mathbb{C}$ be the complex plane. Let $U \subseteq X$, a non-empty open subset of X and let $\mathcal{F}(U)$ be the ring of constant (complex-valued) functions on U, i.e. $\mathcal{F}(U) \cong \mathbb{C}$ for all U. Let the restriction maps $\rho_{U,V}$ for all $V \subseteq U$ be the restriction of the function on U to V. Then $\mathcal{F}$ is a presheaf, but not a sheaf. This is because being constant is not a local property. For example, let $U = U_1 \cup U_2$, where U_1 and U_2 are open discs in $\mathbb{C}$ and $U_1 \cap U_2 = \emptyset$. Let $f_1 : U_1 \to \mathbb{C}$ be the constant function 0, and let $f_2 : U_2 \to \mathbb{C}$ be the constant function 1. Then f_1 and f_2 trivially agree on the overlap $U_1 \cap U_2 = \emptyset$, but there is no constant function on U that restricts to both f_1 and f_2 on U_1 and U_2, respectively.

Example 1.3.5. Let X be a complex manifold, and $U \subseteq X$ an open subset. Let $\mathcal{O}_X(U)$ be the totality of holomorphic functions on U. The restriction map is the restriction of the domain of a function. Then as in Example 1.3.3, $\mathcal{O}_X$ becomes a sheaf of commutative rings over X. The sheaf $\mathcal{O}_X$ is called the *structure sheaf* on X.

Now, if $\mathcal{O}_X$ is any sheaf of rings on a complex manifold X, we introduce the notion of a sheaf of modules over $\mathcal{O}_X$ as follows.

Definition 1.3.6. Let X be a complex manifold, and $\mathcal{O}_X$ be a sheaf of rings on X. A presheaf of abelian groups, $\mathcal{F}$, is called a *sheaf of $\mathcal{O}_X$-modules*, if for every open subset $U \subseteq X$, $\mathcal{F}(U)$ is equipped with the structure of an $\mathcal{O}_X(U)$-module such that for every inclusion $V \subseteq U$, the restriction map $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ is $\mathcal{O}_X(U)$-linear via the ring homomorphism $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$.

Now, we know that it is important to study group homomorphisms between groups, ring homomorphisms between rings and so on. Thus, it is natural now to study morphisms between sheaves, given by the following definition.

Definition 1.3.7. Let X be a complex manifold. If $\mathcal{F}$ and $\mathcal{G}$ are (pre)sheaves of abelian groups on X, a *morphism* $\varphi : \mathcal{F} \to \mathcal{G}$ consists of a collection of homomorphisms $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set $U \subseteq X$, such that whenever $V \subseteq U$ is an inclusion, the diagram

$$\begin{tikzcd} \mathcal{F}(U) \arrow[r, "\varphi(U)"] \arrow[d, "\rho_{U,V}"'] & \mathcal{G}(U) \arrow[d, "\rho'_{U,V}"] \\ \mathcal{F}(V) \arrow[r, "\varphi(V)"] & \mathcal{G}(V) \end{tikzcd}$$

is commutative, where ρ and ρ' are the restriction maps in $\mathcal{F}$ and $\mathcal{G}$, respectively. A (pre)sheaf morphism of abelian groups $\varphi : \mathcal{F} \to \mathcal{G}$ is said to be an *isomorphism* if for each open subset $U \subseteq X$, $\varphi(U)$ is an isomorphism.

Remark 1.3.8. We also have the notion of a morphism of (pre)sheaves of vector spaces (or rings), $\varphi : \mathcal{F} \to \mathcal{G}$, on X which consists of a collection of linear maps (or ring homomorphisms) $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open set $U \subseteq X$. Similarly, a morphism of sheaves of $\mathcal{O}_X$-modules $\varphi : \mathcal{F} \to \mathcal{G}$ on X consists of a collection of homomorphisms of $\mathcal{O}_X(U)$-modules $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open set $U \subseteq X$.

Once we have a morphism of $\varphi : \mathcal{F} \to \mathcal{G}$ of (pre)sheaves of abelian groups we can construct the presheaf kernel, denoted $\ker(\varphi)$, the presheaf image, denoted $\operatorname{im}'(\varphi)$, and the presheaf cokernel, denoted $\operatorname{coker}'(\varphi)$, which are defined in the obvious way: e.g. $\operatorname{coker}'(\varphi)(U) = \operatorname{coker}(\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U))$. It is important to note here that if φ is a sheaf morphism, then $\ker(\varphi)$ is itself a sheaf. However, $\operatorname{im}'(\varphi)$ and $\operatorname{coker}'(\varphi)$ are just presheaves. In order to define the cokernel and image of a sheaf morphism as sheaves, we need to introduce the notion of a stalk.

Definition 1.3.9. Let $\mathcal{F}$ be a (pre)sheaf of abelian groups on X and $x \in X$. The *stalk* of $\mathcal{F}$ at x is

$$\mathcal{F}_x := \{(U, s) | x \in U \subset X, s \in \mathcal{F}(U)\} / \sim .$$

Here, for two open subsets $U_i, i = 1, 2$ and sections $s_i \in \mathcal{F}(U_i), i = 1, 2$, one sets $(U_1, s_1) \sim (U_2, s_2)$ if there exists an open subset $x \in U \subset U_1 \cap U_2$ such that $\rho_{U_1,U}(s_1) = \rho_{U_2,U}(s_2)$. The elements of $\mathcal{F}_x$ are called the *germs* of $\mathcal{F}$.

One immediately finds that any section $s \in \mathcal{F}(U)$ induces an element $s_x \in \mathcal{F}_x$ for any point $x \in U$. Furthermore, any (pre)sheaf morphism $\varphi : \mathcal{F} \to \mathcal{G}$ induces morphisms $\mathcal{F}_x \to \mathcal{G}_x$ for any $x \in X$.

Example 1.3.10. Consider the structure sheaf $\mathcal{O}_X$ from Example 1.3.5 again. If $x \in X$, the stalk of $\mathcal{O}_X$ at x, denoted $\mathcal{O}_{X,x}$, is the ring of germs of holomorphic functions at x.

Definition 1.3.11. Let $\mathcal{F}'$ be a presheaf of abelian groups on a complex manifold, X. The *sheafification* of $\mathcal{F}'$, or the sheaf *associated to* the presheaf $\mathcal{F}'$, is defined to be the sheaf $\mathcal{F}$ for which $\mathcal{F}(U)$, of an open subset $U \subseteq X$, is the set of all maps $s : U \to \bigcup_{x \in U} \mathcal{F}'_x$ with $s(x) \in \mathcal{F}'_x$ and such that for all $x \in U$ there exists an open subset $x \in V \subseteq U$ and a section $t \in \mathcal{F}'(V)$ with $s(y) = t_y$ for all $y \in V$.

With this definition, we show that $\mathcal{F}$ really is a sheaf. It is clear that $\mathcal{F}$ is a presheaf. It remains to show that it satisfies the glueing property, i.e. if $U \subset X$ is an open subset of X, $\{U_i\}$ is an open cover of U and $s_i \in \mathcal{F}(U_i)$ for all i (where $s_i : U_i \to \bigcup_{x \in U_i} \mathcal{F}'_x$) such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, we must show that there is a unique $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all i.

Now for all $x \in U$, define $s(x) \in \mathcal{F}'_x$ to be $s_i(x)$ if $x \in U_i$. If $x \in U_i \cap U_j$, we know $s_i(x) = s_j(x) \in \mathcal{F}'_x$ so the definition is independent of the choice of i. If $x \in U_i$, we have $s(x) = s_i(x)$, i.e. $s|_{U_i} = s_i$. We must now show that this $s \in \mathcal{F}(U)$ is unique, i.e. if $s|_{U_i} = 0$ for all i, then $s = 0$. If $x \in U$ then there exists an i with $x \in U_i$ and $s(x) = s|_{U_i}(x) = 0$ by assumption. Hence $s = 0$ and $\mathcal{F}$ is really a sheaf.

Definition 1.3.12. A *subsheaf* of a sheaf of abelian groups, $\mathcal{F}$, is a sheaf of abelian groups $\mathcal{F}'$ such that for every open set $U \subseteq X$, $\mathcal{F}'(U)$ is a subgroup of $\mathcal{F}(U)$, and the restriction maps $\rho'_{U,V} : \mathcal{F}'(U) \to \mathcal{F}'(V)$ of the sheaf $\mathcal{F}'$ are induced by those of $\mathcal{F}$, i.e. they are obtained by restricting $\rho_{U,V} : \mathcal{F}(U) \to \mathcal{F}(V)$ to $\mathcal{F}'(U)$. It follows that for any point x, the stalk $\mathcal{F}'_x$ is a subgroup of $\mathcal{F}_x$.

Remark 1.3.13. We also have the notion of a subsheaf of a sheaf of vector spaces $\mathcal{F}$. A subsheaf of $\mathcal{F}$ is a sheaf of vector spaces $\mathcal{F}'$ such that $\mathcal{F}'(U)$ is a subvector space of $\mathcal{F}(U)$. Similarly, we can define a subsheaf of a sheaf of rings $\mathcal{F}$ as a sheaf of rings $\mathcal{F}'$ such that $\mathcal{F}'(U)$ is a subring of $\mathcal{F}(U)$.

Using sheafification, we can now give the following definitions:

Definition 1.3.14. (a) If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups, we define the *cokernel* of φ, denoted $\operatorname{coker}(\varphi)$, to be the sheaf associated to the presheaf $\operatorname{coker}'(\varphi)$.

(b) The morphism φ is called a *monomorphism* if $\ker(\varphi) = 0$.

(c) If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups, we define the *image* of φ, denote $\operatorname{im}(\varphi)$, to be the sheaf associated to the presheaf $\operatorname{im}'(\varphi)$.

(d) The morphism φ is called an *epimorphism* if $\operatorname{im}(\varphi) = \mathcal{G}$.

(e) If the morphism φ is a monomorphism, its cokernel is denoted $\mathcal{G}/\mathcal{F}$ and called the *quotient* of $\mathcal{G}$ by $\mathcal{F}$.

(f) We say that a sequence

$$\cdots \longrightarrow \mathcal{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathcal{F}^{i} \xrightarrow{\varphi^{i}} \mathcal{F}^{i+1} \longrightarrow \cdots$$

of sheaves and morphisms is *exact* if at each stage $\ker(\varphi^i) = \operatorname{im}(\varphi^{i-1})$.

It can be shown that these definitions of the kernel and cokernel of a morphism of sheaves satisfy the universal property which is used in category theory (Definitions 1.4.7 and 1.4.8).

Remark 1.3.15. If X is a complex manifold, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of sheaves is a monomorphism if and only if the map on sections $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is a monomorphism for each $U \subseteq X$, i.e. $\ker(\varphi) = 0$ if and only if $\ker(\varphi(U)) = 0$ for each U.

The corresponding statement for epimorphisms is not true: if $\varphi : \mathcal{F} \to \mathcal{G}$ is an epimorphism, the maps $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ on sections need not be epimorphisms. See Example 2.1.17 to illustrate this fact.

1.3.1 Coherent sheaves

It turns out that sheaves of modules are too general objects for many applications. Coherent sheaves are a special class of sheaves with particularly manageable properties. Many results and properties in algebraic geometry are formulated in terms of coherent sheaves and their cohomology.

Definition 1.3.16. Let X be a complex manifold. A sheaf of $\mathcal{O}_X$-modules, $\mathcal{F}$, is a *coherent sheaf* if and only if

(a) for each $x \in X$, there exists and open set $U \subseteq X$ with $x \in U$ and there exist an integer n such that there exists an exact sequence of sheaves on U

$$\mathcal{O}_X^n|_U \to \mathcal{F}|_U \to 0,$$

where $\mathcal{O}_X^n := \bigoplus_1^n \mathcal{O}_X$ (In this case, we say $\mathcal{F}$ is a *finitely generated* $\mathcal{O}_X$-module) and

(b) For any open set $U \subseteq X$ and for any $\mathcal{O}_X|_U$-module homomorphism

$$\varphi : \mathcal{O}_X^n|_U \to \mathcal{F}|_U,$$

$\ker(\varphi)$ is a finitely generated $\mathcal{O}_X$-module.

Example 1.3.17. By the Theorem of Oka, the structure sheaf $\mathcal{O}_X$ on a complex manifold, X, is a coherent sheaf.

The following is an example of a noncoherent sheaf.

Example 1.3.18. Let V be an infinite dimensional complex vector space and let $x \in X$ be a fixed point. Define

$$\mathcal{F}(U) := \begin{cases} V & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

The $\mathcal{O}_X(U)$-module structure is then given by the following: if $x \in U$, $f \in \mathcal{O}_X(U)$ and $v \in V = \mathcal{F}(U)$, we define $f \cdot v := f(x) \cdot v$. Since $\mathcal{F}$ is not a finitely generated $\mathcal{O}_X$-module, it is noncoherent.

Remark 1.3.19. If $\mathcal{F}$ and $\mathcal{G}$ are coherent sheaves of $\mathcal{O}_X$-modules and $\varphi : \mathcal{F} \to \mathcal{G}$ a morphism of sheaves, then $\ker(\varphi), \operatorname{coker}(\varphi), \operatorname{im}(\varphi)$ are coherent sheaves of $\mathcal{O}_X$-modules.

Definition 1.3.20. A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a complex manifold X is called *locally free* of rank r if it is locally isomorphic to the sheaf $\mathcal{O}_X^r := \bigoplus_1^r \mathcal{O}_X$, i.e. if there exists an open cover $\{U_i\}_{i \in \mathcal{I}}$ of X such that $\mathcal{F}|_{U_i} \cong \mathcal{O}_X^r|_{U_i}$. When $r = 1$, the sheaf is called *invertible*.

Definition 1.3.21. Let C be a smooth projective curve with structure sheaf $\mathcal{O}_C$. If $\mathcal{F}$ is a sheaf of $\mathcal{O}_C$-modules, we define the *torsion subsheaf*, $\operatorname{tors}(\mathcal{F})$, of $\mathcal{F}$ as follows: For any open subset $U \subseteq C$ we let

$$\operatorname{tors}(\mathcal{F})(U) := \{s \in \mathcal{F}(U) | \exists f \in \mathcal{O}_C(U) \text{ such that } f \neq 0 \text{ and } f \cdot s = 0\}.$$

Briefly, $\operatorname{tors}(\mathcal{F})(U)$ is the torsion sub-module of the $\mathcal{O}_C(U)$-module, $\mathcal{F}(U)$. The presheaf, $\operatorname{tors}(\mathcal{F})$, is actually a sheaf. A sheaf, $\mathcal{F}$, is called *torsion free* if $\operatorname{tors}(\mathcal{F}) = 0$ and it's called a *torsion sheaf* if $\operatorname{tors}(\mathcal{F}) = \mathcal{F}$.

Remark 1.3.22. We know from [OSS] Chapter II, 1.1, that a finitely generated module over a regular local ring of dimension one is locally free if and only if it is torsion free. Therefore, on a smooth curve C, a coherent sheaf $\mathcal{F}$ is locally free if and only if $\operatorname{tors}(\mathcal{F}) = 0$.

In particular, for any coherent sheaf $\mathcal{F}$, $\mathcal{F}/\operatorname{tors}(\mathcal{F})$ is locally free. If $\mathcal{F}' \subset \mathcal{F}$ is a subsheaf, we also have $\operatorname{tors}(\mathcal{F}') \subset \operatorname{tors}(\mathcal{F})$. In particular, if $\mathcal{F}$ is torsion free then any subsheaf $\mathcal{F}'$ is torsion free.

1.3.2 Čech cohomology

All the definitions and formulations made in this subsection will be made on a projective complex manifold. However, everything here can also be defined on a general topological space.

Motivation for cohomology

The main basis of the motivation of the theory of cohomology of sheaves is based on the fact that the functor (see Section 1.4.1 for the definition of a functor) taking global sections of a sheaf is not exact but only left exact, i.e. given an exact sequence of sheaves of abelian groups

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

on a projective complex manifold X, by taking global sections we get an exact sequence

$$0 \to \Gamma(\mathcal{F}_1) \to \Gamma(\mathcal{F}_2) \to \Gamma(\mathcal{F}_3)$$

of abelian groups in which the last map $\Gamma(\mathcal{F}_2) \to \Gamma(\mathcal{F}_3)$ is in general not surjective. Example 2.1.17 illustrates this.

The goal of cohomology is to extend the global section sequence to the right in the following sense: for any sheaf $\mathcal{F}$ of abelian groups on X we will define cohomology groups $H^i(X, \mathcal{F})$, for all $i > 0$ satisfying (among other things) the following property: given any exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

of sheaves on X, there is an induced long exact sequence of cohomology groups

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(X, \mathcal{F}_1) & \longrightarrow & \Gamma(X, \mathcal{F}_2) & \longrightarrow & \Gamma(X, \mathcal{F}_3) & \longrightarrow & H^1(X, \mathcal{F}_1) \\ & \longrightarrow & H^1(X, \mathcal{F}_2) & \longrightarrow & H^1(X, \mathcal{F}_3) & \longrightarrow & H^2(X, \mathcal{F}_1) & \longrightarrow & \cdots \end{array}$$

where $\Gamma(X, \mathcal{F}_1) = H^0(X, \mathcal{F}_1)$. Let us now give the defintion of these cohomology groups. There are many ways to define these groups but the approach we will use here is the approach of Čech cohomology. The idea of Čech cohomology is simple: If X is a projective complex manifold, we choose an open cover $U = \{U_i\}$ of X and consider sections of our sheaves on these open subsets and their intersections.

Definition 1.3.23. Let X be a complex manifold, and let $\mathcal{F}$ be a sheaf on X. Fix an open cover $\{U_i\}_{i\in I}$ of X and assume that I is an ordered set. For all $p \geq 0$ we define the abelian group

$$C^p(\mathcal{F}) := \prod_{i_0<\ldots<i_p} \mathcal{F}(U_{i_0} \cap \ldots \cap U_{i_p}).$$

In other words, an element $s \in C^p(\mathcal{F})$ is a collection $s = (s_{i_0,\ldots,i_p})$ of sections of $\mathcal{F}$ (which can be totally unrelated) over all intersections of $p+1$ sets taken from the cover.

For every $p \geq 0$ we define a *boundary operator* $d^p : C^p(\mathcal{F}) \to C^{p+1}(\mathcal{F})$ by

$$(d^p s)_{i_0,\ldots,i_{p+1}} = \sum\nolimits_{k=0}^{p+1} (-1)^k s_{i_0,\ldots,i_{k-1},i_{k+1},\ldots,i_{p+1}}|_{U_{i_0}\cap\ldots\cap U_{i_{p+1}}}.$$

Note that this makes sense since the $s_{i_0,\ldots,i_{k-1},i_{k+1},i_{p+1}}$ are sections of $\mathcal{F}$ on $U_{i_0} \cap \ldots \cap U_{i_{k-1}} \cap U_{i_{k+1}} \cap \ldots \cap U_{i_{p+1}}$, which contains $U_{i_0} \cap \ldots \cap U_{i_{p+1}}$ as an open subset.

By abuse of notation we will denote all these operators simply by d if it is clear from the context on which $C^p(\mathcal{F})$ they act.

Lemma 1.3.24. *Let $\mathcal{F}$ be a sheaf of abelian groups on a complex manifold X. Then $d^{p+1} \circ d^p : C^p(\mathcal{F}) \to C^{p+2}(\mathcal{F})$ is the zero map for all $p \geq 0$.*

Proof. ([G] Lemma 8.1.3) For every $s \in C^p(\mathcal{F})$ we have

$$(d^{p+1}(d^p s))_{i_0,\ldots,i_{p+2}} = \sum_{k=0}^{p+2} (-1)^k (ds)_{i_0,\ldots,i_{k-1},i_{k+1},\ldots,i_{p+2}}$$

From the definition of ds, we then get:

$$\begin{aligned}(d^{p+1}(d^p s))_{i_0,\ldots,i_{p+2}} &= \sum_{k=0}^{p+2}\sum_{m=0}^{k-1} (-1)^{k+m} s_{i_0,\ldots,i_{m-1},i_{m+1},\ldots,i_{k-1},i_{k+1},\ldots,i_{p+2}} \\ &\quad + \sum_{k=0}^{p+2}\sum_{m=k+1}^{p+2} (-1)^{k+m-1} s_{i_0,\ldots,i_{k-1},i_{k+1},\ldots,i_{m-1},i_{m+1},\ldots,i_{p+2}} \\ &= 0\end{aligned}$$

Note that the restriction maps are ommitted from the above equations for ease of reading. $\square$

We have thus defined a sequence of abelian groups and homomorphisms

$$C^0(\mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{F}) \xrightarrow{d^1} C^2(\mathcal{F}) \xrightarrow{d^2} \dots$$

such that $d^{p+1} \circ d^p = 0$ at every step. Such a sequence is called a *complex* of abelian groups, with the maps d^p being the boundary operators.

Definition 1.3.25. Let $\mathcal{F}$ be a sheaf of abelian groups on a complex manifold X. Pick an open cover $\{U_i\}$ of X and consider the associated groups $C^p(\mathcal{F})$ and homomorphisms $d^p : C^p(\mathcal{F}) \to C^{p+1}(\mathcal{F})$ for $p \geq 0$. We define the p-th *cohomology group* of $\mathcal{F}$ to be

$$H^p(X, \mathcal{F}) = \ker d^p / \operatorname{im} d^{p-1}$$

with the convention that $C^p(\mathcal{F})$ and d^p are zero for $p < 0$. Note that this is well-defined as $\operatorname{im} d^{p-1} \subset \ker d^p$ by Lemma 1.3.24.

Remark 1.3.26. Note that this defintion of the cohomology groups depends on the choice of the open cover of X. However, if the open cover is chosen appropriately, this dependence disappears. There are other constructions of the cohomology groups, such as the derived functor approach, that do not depend on the choice of an open cover and therefore do not face this problem.

For our purpose, it is sufficient to know the following: If X is a complex projective manifold and $X \subset \mathbb{P}^n$ an embedding, then let $U_i = \{z_i \neq 0\} \subset \mathbb{P}^n$ be the standard open subsets of $\mathbb{P}^n$. The sets $X \cap U_i$ form an open cover of X. The Čech cohomology groups obtained using this open cover then coincide with the cohomology groups one would obtain using the derived functor approach and therefore the dependence on the open cover disappears.

1.4 Categories

Categories were first introduced by Samuel Eilenberg and Saunders Mac Lane in 1942-1945, in connection with algebraic topology. The study of categories is known as category theory. It deals in an abstract way with mathematical structures and relationships between them. Categories are particularly useful in this thesis to provide us with a central unifying notion.

Definition 1.4.1. A *category* $\mathcal{C}$ consists of the following: a class $\mathrm{obj}(\mathcal{C})$ of *objects*, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of *morphisms* for every ordered pair of objects, (A, B), an *identity morphism* $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ for each object A, and a *composition function* $\mathrm{Hom}_{\mathcal{C}}(A, B) \times \mathrm{Hom}_{\mathcal{C}}(B, C) \to \mathrm{Hom}_{\mathcal{C}}(A, C)$ for every ordered triple (A, B, C) of objects. We write $f : A \to B$ to indicate that f is a morphism in $\mathrm{Hom}_{\mathcal{C}}(A, B)$, and we denote the composition of $f : A \to B$ with $g : B \to C$ by gf or $g \circ f$. The above data is subject to two axioms:
Associativity Axiom:

$$(hg)f = h(gf), \text{ for all } f : A \to B, g : B \to C, h : C \to D \text{ in } \mathcal{C}$$

Unit Axiom:

$$\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A, \text{ for all } f : A \to B.$$

Example 1.4.2. One example to keep in mind is the category **Sets** of sets. The objects are sets and the morphisms are (set) maps, that is, the elements of $\mathrm{Hom}_{\mathbf{Sets}}(A, B)$ are the maps from A to B. Composition of morphisms is just composition of maps, and id_A is the map $\mathrm{id}_A(a) = a$ for all $a \in A$.

One can similarly define the category **Groups** in which the objects are groups and the morphisms are group homomorphisms. Composition is just composition of group homomorphisms. The category **Rings**, is the category whose objects are commutative rings with unity and the morphisms are ring homomorphisms. The category **Ab** has objects abelian groups and the morphisms are group homomorphisms.

Definition 1.4.3. A morphism $f : A \to B$ in a category, $\mathcal{C}$ is called an *isomorphism* if there is a morphism $g : B \to A$ such that $gf = \mathrm{id}_A$ and $fg = \mathrm{id}_B$. If such a morphism exists, we say that A is isomorphic to B, denoted $A \cong B$. An isomorphism in **Sets** is a set bijection.

Definition 1.4.4. A morphism $f : B \to C$ is called a *monomorphism* in $\mathcal{C}$ if for all objects A in $\mathcal{C}$ and any two distinct morphisms $e_1, e_2 : A \to B$ we have $fe_1 \neq fe_2$. In **Sets** and **Ab**, in which objects have an underlying set, the monomorphisms are precisely the morphisms that are set injections in the usual sense.

A morphism $f : B \to C$ is called an *epimorphism* in $\mathcal{C}$ if for all objects $D \in \mathcal{C}$ and any two distinct morphisms $g_1, g_2 : C \to D$ we have $g_1 f \neq g_2 f$. In **Sets** and **Ab**, the epimorphisms are precisely the surjective morphisms.

Definition 1.4.5. A *zero object* is an object with precisely one morphism to and from each object. We reserve the symbol 0 for the zero object. The category **Sets** does not have a zero object, while the category **Groups** does (namely the group with one element).

Let $\mathcal{C}$ be a category with a zero object and let $A, B \in \mathcal{C}$. The unique morphism $A \to 0 \to B$ is called the *zero morphism*, denoted $0 \in \mathrm{Hom}_{\mathcal{C}}(A, B)$.

Definition 1.4.6. Two monomorphisms $A_1 \to B$ and $A_2 \to B$ are *equivalent* if there are morphisms $A_1 \to A_2$ and $A_2 \to A_1$ such that the following two diagrams commute

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d] & B \\ A_2 \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[r] & B \\ A_2 \arrow[u] \arrow[ur] & \end{tikzcd}$$

A *subobject* of B is an equivalence class of monomorphisms into B.

Similarly two epimorphisms $B \to C_1$ and $B \to C_2$ are *equivalent* if there are morphisms $C_1 \to C_2$ and $C_2 \to C_1$ such that the following two diagrams commute

$$\begin{tikzcd} B \arrow[r] \arrow[dr] & C_1 \arrow[d] \\ & C_2 \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r] \arrow[dr] & C_1 \\ & C_2 \arrow[u] \end{tikzcd}$$

A *quotient object* is an equivalence class of epimorphisms.

Definition 1.4.7. Let $\mathcal{C}$ be a category with a zero object, 0. A *kernel* of a morphism $f : B \to C$ is a morphism $i : A \to B$ such that $fi = 0$ and that satisfies the following universal property: Every morphism $e : A' \to B$ in $\mathcal{C}$ such that $fe = 0$ factors uniquely through A as $e = ie'$ for a unique $e' : A' \to A$. This can be visualised by the following commutative diagram:

$$\begin{tikzcd} A \arrow[r, "i"] & B \arrow[r, "f"] & C \\ A' \arrow[u, dashed, "e'"] \arrow[ur, "e"'] & & \end{tikzcd}$$

Every kernel is a monomorphism, and two kernels of f are isomorphic; we often identify the kernel of f with the corresponding subobject of B, and denote it $\ker(f)$.

Definition 1.4.8. Let $\mathcal{C}$ be a category with a zero object. A *cokernel* of a morphism $f : B \to C$ is a morphism $p : C \to D$ such that $pf = 0$ and that satisfies the following universal property: Every morphism $g : C \to D'$ such that $gf = 0$ factors uniquely through D as $g = g'p$ for a unique $g' : D \to D'$.

$$\begin{tikzcd} B \arrow[r, "f"] & C \arrow[r, "p"] \arrow[dr, "g"'] & D \arrow[d, dashed, "g'"] \\ & & D' \end{tikzcd}$$

We often identify the cokernel of f with the corresponding object, D, and denote it $\mathrm{coker}(f)$. Every cokernel is an epimorphism, and any two cokernels are isomorphic. In **Ab** and **Groups** kernel and cokernel have their usual meanings.

Definition 1.4.9. Let $\mathcal{C}$ be a category which admits kernels and cokernels (i.e. every morphism has a kernel and a cokernel). Let $f : A \to B$ be a morphism in $\mathcal{C}$. We define the *coimage* of f, denoted $\mathrm{coim}(f)$, to be the cokernel of h, where $h : \ker f \to A$. We define the *image* of f, denoted $\mathrm{im}(f)$ to be the kernel of g, where $g : B \to \mathrm{coker}(f)$. Consider now the following diagram:

$$\begin{tikzcd}
\ker(f) \arrow[r, "h"] & A \arrow[r, "f"] \arrow[d, "k"] & B \arrow[r, "g"] & \operatorname{coker}(f) \\
& \operatorname{coim}(f) \arrow[r, dashed, "u"] \arrow[ur, dashed, "\overline{f}"] & \operatorname{im}(f) \arrow[u] &
\end{tikzcd}$$

Now since $fh = 0$, f factors uniquely through $\overline{f}k$ as $f = \overline{f} \circ k$. By definition of cokernel, $g \circ f = g \circ \overline{f} \circ k = 0$ and k is an epimorphism (as it is a cokernel). Hence we get $g \circ \overline{f} = 0$. Since $\operatorname{im}(f)$ is the kernel of g, by the universal property of the kernel, we know that there exists a unique morphism $u : \operatorname{coim}(f) \to \operatorname{im}(f)$ such that the diagram above is commutative.

Definition 1.4.10. Let $\mathcal{C}$ be a category which admits kernels and cokernels, and let $f : A \to B$ be a morphism in $\mathcal{C}$. The morphism f is called *strict* if $u : \operatorname{coim}(f) \to \operatorname{im}(f)$ constructed above is an isomorphism.

Definition 1.4.11. Let $\mathcal{C}$ be a category which admits kernels and cokernels, and let C_i be objects in $\mathcal{C}$ for all i. A sequence of morphisms in $\mathcal{C}$

$$\cdots \longrightarrow C_1 \xrightarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xrightarrow{f_3} \cdots$$

is *exact* if $\operatorname{im}(f_{i-1}) = \ker(f_i)$ for all i. If A, B, C are objects in $\mathcal{C}$ we say a sequence of morphisms in $\mathcal{C}$

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a *short exact sequence* if $\operatorname{im}(f) = \ker(g)$ and f is a monomorphism and g is an epimorphism.

Definition 1.4.12. An object P is called a *product* of two objects A and B if there exist morphisms $P \xrightarrow{p_1} A$ and $P \xrightarrow{p_2} B$ such that for every pair of morphisms $X \to A$ and $X \to B$ there is a unique $X \to P$ such that

$$\begin{tikzcd}
& A \\
X \arrow[ur] \arrow[r, dashed] \arrow[dr] & P \arrow[u, "p_1"'] \arrow[d, "pr_2"] \\
& B
\end{tikzcd}$$

commutes.

In the categories, **Sets**, **Groups** and **Rings** products can be constructed by taking Cartesian products.

Proposition 1.4.13. *If P and P' are products of A and B they are isomorphic.*

Proof. ([F] Proposition 1.71) Let $P \xrightarrow{p_1} A$, $P \xrightarrow{p_2} B$, $P' \xrightarrow{p_1'} A$, and $P' \xrightarrow{p_2'} B$, be the morphisms satisfying the conditions of the definition of a product of A and B. According to the definition we know there is a unique morphism $P \to P'$ such that the following diagram

$$\begin{tikzcd} & A \\ P \arrow[ur, "p_1"] \arrow[r, "f"] \arrow[dr, "p_2"'] & P' \arrow[u, "p_1'"'] \arrow[d, "p_2'"] \\ & B \end{tikzcd}$$

commutes and there is a unique morphism $P' \to P$ such that the diagram

$$\begin{tikzcd} & A \\ P' \arrow[ur, "p_1'"] \arrow[r, "g"] \arrow[dr, "p_2'"'] & P \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & B \end{tikzcd}$$

commutes. The composition $x := g \circ f$ gives us the following commutative diagram

$$\begin{tikzcd} & A \\ P \arrow[ur, "p_1"] \arrow[r, "x"] \arrow[dr, "p_2"'] & P \arrow[u, "p_1"'] \arrow[d, "p_2"] \\ & B \end{tikzcd}$$

By definition of product, we know that x must be unique and this implies that $x = \mathrm{id}_P$, the identity morphism of P. Similarly we can show that $f \circ g = \mathrm{id}_{P'}$. $\square$

The dual of the product is the sum. The definition of sum is just the definition of product with all arrows reversed as follows:

Definition 1.4.14. An object S is called a *sum* of two objects, A and B, if there exist morphisms $A \xrightarrow{\mu_1} S$ and $B \xrightarrow{\mu_2} S$ such that for every pair of morphisms $A \to X$ and $B \to X$ there is a unique $S \to X$ such that

$$\begin{tikzcd} A \arrow[d, "\mu_1"'] \arrow[dr] & \\ S \arrow[r, dashed] & X \\ B \arrow[u, "\mu_2"] \arrow[ur] & \end{tikzcd}$$

commutes.

Proposition 1.4.15. *If S and S' are sums of A and B, they are isomorphic.*

Proof. The proof is analogous to the proof of 1.4.13 with arrows reversed. $\square$

In well-known categories the word "sum" is traditionally replaced by:

Category	Sum
Sets	Disjoint union
Ab	Direct sum (Cartesian product)
Groups	Free product
Rings	Tensor product

Remark 1.4.16. The product and the sum do not always exist in categories. In Chapter 4 we will see an example of a category ($\mathbf{CohSys}^{\mathbf{st}}(X)$) in which the product of two objects does not exist.

Definition 1.4.17. A category $\mathcal{C}$ is called of *finite length* if it satisfies the following two conditions:

(a) Any sequence of epimorphisms stabilises, i.e. for any sequence of epimorphisms $f_k : C_k \to C_{k+1}$ there exists an integer k_0 such that f_k is an isomorphism for all $k \geq k_0$.

(b) Any sequence of monomorphisms stabilises, i.e. for any sequence of monomorphisms $g_k : C_k \to C_{k+1}$ there exists an integer k_0 such that g_k is an isomorphism for all $k \leq k_0$.

Example 1.4.18. The category of finite dimensional vector spaces over a fixed field is of finite length. This is a consequence of the fact that a linear map between finite dimensional vector spaces of the same dimension which is either an epimorphism or a monomorphism is automatically an isomorphism. This is no longer true for vector spaces of infinite dimension. In fact it is not hard to give examples which show that the category of (not necessarily finite dimensional) vector spaces neither satisfies (a) nor (b) in Definition 1.4.17.

1.4.1 Functors

We have seen that a category is itself a mathematical structure. Let us now look at a 'process', namely a functor, which preserves this structure in some precise sense, outlined below.

Definition 1.4.19. A *functor* $F : \mathcal{C} \to \mathcal{D}$ from a category $\mathcal{C}$ to a category $\mathcal{D}$ associates an object $F(C)$ of $\mathcal{D}$ to every object C of $\mathcal{C}$, and a morphism $F(f) : F(C_1) \to F(C_2)$ in $\mathcal{D}$ to every morphism $f : C_1 \to C_2$ in $\mathcal{C}$. We require F to preserve identity morphisms ($F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$) and composition ($F(gf) = F(g)F(f)$). Note that F induces set maps $\mathrm{Hom}_{\mathcal{C}}(C_1, C_2) \to \mathrm{Hom}_{\mathcal{D}}(F(C_1), F(C_2))$, for every C_1, C_2 in $\mathcal{C}$.

The identity functor $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ fixes all objects and morphisms, that is, $\mathrm{id}_{\mathcal{C}}(C) = C, \mathrm{id}_{\mathcal{C}}(f) = f$. Clearly, for a functor $F : \mathcal{C} \to \mathcal{D}$ we have $F \circ \mathrm{id}_{\mathcal{C}} = F = \mathrm{id}_{\mathcal{D}} \circ F$.

Example 1.4.20. We have seen an example of a functor in Section 1.3, namely the global sections functor. This is a functor from the category, $\mathbf{Sh}(X)$, of sheaves of abelian groups (with sheaf morphisms) on a complex manifold, X, to $\mathbf{Ab}$.

Example 1.4.21. Let $\mathcal{C}$ be any category and let N be a fixed object in $\mathcal{C}$. Consider a functor $F := \mathrm{Hom}(N, -)$ defined by $F(A) = \mathrm{Hom}_{\mathcal{C}}(N, A)$ for any object $A \in \mathcal{C}$ and for any morphism $f : A \to B$, then $F(f) : \mathrm{Hom}(N, A) \to \mathrm{Hom}(N, B)$ is defined by $F(f)(\alpha) := f \circ \alpha$, for all $\alpha \in \mathrm{Hom}(N, A)$. Let us check that this really is a functor. Clearly F preserves identity, i.e. $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$. Let $f : A \to B$ and $g : B \to C$ be two morphisms in $\mathcal{C}$. We must check that $F(gf) = F(g)F(f)$ so we have

$$F(gf)(\alpha) = g \circ f \circ \alpha$$

and

$$F(g)(F(f)(\alpha)) = F(g)(f \circ \alpha) = g \circ f \circ \alpha.$$

So we see that F preserves composition. Hence, $F = \mathrm{Hom}_{\mathcal{C}}(N, -)$ is a functor from $\mathcal{C}$ to $\mathbf{Sets}$.

Definition 1.4.22. A funtor, $F : \mathcal{C} \to \mathcal{D}$ is called *left exact* (resp. *right exact*) if for every short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$, the seqence $0 \to F(A) \to F(B) \to F(C)$ (resp. $F(A) \to F(B) \to F(C) \to 0$) is exact in $\mathcal{D}$. F is called *exact* if it is both left and right exact.

Example 1.4.23. The global section functor of Example 1.4.20 is left exact. We will see another example of a left exact functor in Section 1.4.4, Proposition 1.4.58.

Definition 1.4.24. A functor $F : \mathcal{C} \to \mathcal{D}$ is called *faithful* if the set maps $\mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$ are all injections. That is, if f_1 and f_2 are distinct morphisms from C to C' in $\mathcal{C}$, then $F(f_1) \neq F(f_2)$.

A *subcategory* $\mathcal{B}$ of a category $\mathcal{C}$ is a collection of some of the objects and some of the morphisms, such that the morphisms of $\mathcal{B}$ are closed under composition and include id_B for every $B \in \mathcal{B}$. A subcategory is a category in its own right, and there is an *inclusion functor*, which is faithful by definition.

A subcategory $\mathcal{B}$ in which $\mathrm{Hom}_{\mathcal{B}}(B, B') = \mathrm{Hom}_{\mathcal{C}}(B, B')$ for every B, B' in $\mathcal{B}$ is called a *full subcategory*. A full subcategory is called a *strictly full subcategory* if for any $B \in \mathcal{B}$ and $C \in \mathcal{C}$ with $B \cong C$, we have $C \in \mathcal{B}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is called *full* if $\mathrm{Hom}_{\mathcal{C}}(C, C') \to \mathrm{Hom}_{\mathcal{D}}(F(C), F(C'))$ are all surjections. That is, every $g : F(C) \to F(C')$ in $\mathcal{D}$ is of the form $g = F(f)$ for some $f : C \to C'$. A functor that is both full and faithful is called *fully faithful*.

All the functors above are called *covariant* functors, in order to distinguish them from contravariant functors, defined below.

Definition 1.4.25. A *contravariant* functor $F : \mathcal{C} \to \mathcal{D}$ associates an object $F(C)$ of $\mathcal{D}$ to every object C of $\mathcal{C}$, and a morphism $F(f) : F(C_2) \to F(C_1)$ in $\mathcal{D}$ to every $f : C_1 \to C_2$ in $\mathcal{C}$. Moreover, $F(\mathrm{id}_C) = \mathrm{id}_{F(C)}$ and F reverses composition, i.e. $F(gf) = F(f)F(g)$ for all f, g, composable morphisms in $\mathcal{C}$.

Example 1.4.26. Let $\mathcal{C}$ be any category. Similar to Example 1.4.21 if we let $M \in \mathcal{C}$, then $\mathrm{Hom}_{\mathcal{C}}(-, M)$ is a contravariant functor from $\mathcal{C}$ to **Sets**.

1.4.2 Abelian categories

We need to define an important type of category for the purpose of this thesis, namely an abelian category. The axioms required for a category to be an abelian category are as follows:

Definition 1.4.27. A category $\mathcal{A}$ is *abelian* if

1. $\mathcal{A}$ has a zero object.
2. For every pair of objects there is a product.
2*. For every pair of objects there is a sum.
3. Every morphism has a kernel and a cokernel.
4. Every epimorphism is the cokernel of its kernel.
5. Every monomorphism is the kernel of its cokernel.

Remark 1.4.28. Only one of axioms 2 or 2^* suffices for a category to be abelian, i.e. each in the presence of the other axioms implies the other. The proof of this is not straightforward and can be found in Section 1.598 of [FS]. It is possible to show (See [F], Theorem 2.39) that these axioms suffice to introduce a group structure on each set $\mathrm{Hom}_{\mathcal{A}}(A, B)$, for $A, B \in \mathcal{A}$ in such a way that composition distributes over addition, i.e. given $f, f' : A \to B$ and $g, g' : B \to C$, we have

$$(g + g') \circ (f + f') = g \circ f + g \circ f' + g' \circ f + g' \circ f'.$$

Example 1.4.29. The category **Ab** of abelian groups is an abelian category. Another abelian category is the category $\mathbf{Coh}(X)$ of coherent sheaves on a complex manifold, X.

Definition 1.4.30. An *additive category*, $\mathcal{A}$, is a category in which for any two objects A, B in $\mathcal{A}$ the set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is equipped with the structure of an abelian group such that composition distributes over addition and $\mathcal{A}$ has a zero object and a product $A \times B$ for every pair A, B of objects in $\mathcal{A}$.

Due to Remark 1.4.28, each abelian category is an additive category. However, there exist additive categories which are not abelian.

Example 1.4.31. The category $\mathbf{CohSys}(C)$, of coherent systems with morphisms of coherent systems on a curve C, which will be outlined in Chapter 4 is an additive category but is not an abelian category.

Lemma 1.4.32. *A morphism $f : B \to C$ in an additive category $\mathcal{A}$ is a monomorphism if and only if* $\ker f = 0$.

Proof. Let $f : B \to C$ be a monomorphism in $\mathcal{A}$. By definition, we know that for any $e_1, e_2 : A \to B$, $fe_1 = fe_2$ implies $e_1 = e_2$. We want to show that $\ker(f) = 0$, i.e. given

$$\begin{tikzcd} 0 \arrow[r, "e"] & B \arrow[r, "f"] & C \\ A' \arrow[u, dashed, "g"] \arrow[ur, "e'"'] & & \end{tikzcd}$$

where $fe' = 0$, we must show that there exists a unique g such that $eg = e'$. The only morphism $A' \to 0$ is the zero morphism. It remains to show then that $e' : A' \to B$ is the zero morphism. This follows again from the fact that f is a monomorphism, because $f \circ e' = f \circ 0 = 0$.

Now assume that $\ker(f) = 0$, we want to show that $f : B \to C$ is a monomorphism. Since $\ker(f) = 0$, we have the following diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "f"] & C \\
A' \arrow[u] \arrow[ur, "e'"'] & &
\end{tikzcd}$$

So for any morphism $e' : A' \to B$ such that $fe' = 0$, we have that $e' = 0$. Hence if $e_1, e_2 : A \to B$ are two morphisms such that $fe_1 = fe_2$, then $f \circ (e_1 - e_2) = 0$ and so $e_1 = e_2$. This shows that f is a monomorphism. □

Lemma 1.4.33. *A morphism f in an additive category, $\mathcal{A}$, is an epimorphism if and only if* $\operatorname{coker} f = 0$.

Proof. The proof is analagous to the monomorphism case (Lemma 1.4.32) with arrows reversed. □

Proposition 1.4.34. *([KS] Definition 8.3.5) An additive category, $\mathcal{A}$, is abelian if and only if:*

(a) every morphism has a kernel and cokernel,

(b) any morphism f in $\mathcal{A}$ is strict (See Definition 1.4.10).

Proof. If $\mathcal{A}$ is an abelian category, by definition every morphism has a kernel and cokernel. The proof of (b) is more involved, see [F] Theorem 2.11.

Now assume that $\mathcal{A}$ is an additive category in which every morphism has a kernel and cokernel and any morphism f in $\mathcal{A}$ is strict. We want to show that $\mathcal{A}$ is abelian. Clearly under these assumptions, Axioms 1,2,3 of Definition 1.4.27 are satisfied. So we must now show that every epimorphism is the cokernel of its kernel. Assume $f : A \to B$ is an epimorphism in $\mathcal{A}$. Since f is strict, we know that $\operatorname{im} f \cong \operatorname{coim} f$. We also know, from Lemma

1.4.33, that $\mathrm{coker}(f) = 0$. This gives us that $\mathrm{im}(f) = \ker(B \to 0) = B$. Consider then the following diagram in which u is an isomorphism:

$$\begin{tikzcd}
\ker(f) \arrow[r, "h"] & A \arrow[r, "f"] \arrow[d, "k"'] & B \arrow[r, "g"] \arrow[d, equal] & 0 \\
& \mathrm{coim}(f) \arrow[r, "u"] & B &
\end{tikzcd}$$

By definition $\mathrm{coim}(f) = \mathrm{coker}(h)$. Through the isomorphism u, we then get that the $f : A \to B$ is the cokernel of its kernel.

Similarly we can show that every monomorphism is the kernel of its cokernel. Hence, $\mathcal{A}$ is an abelian category. $\square$

1.4.3 Derived categories

The derived category, $\mathcal{D}(\mathcal{A})$, of an abelian category $\mathcal{A}$ is a construction that was developed in homological algebra by Alexander Grothendieck and his student Jean-Louis Verdier in the 1960's. The construction of a dervied category is carried out using a number of steps which we will outline below. Throughout this section $\mathcal{A}$ will denote an abelian category, unless otherwise specified.

Step 1 :

We define $\mathcal{C}(\mathcal{A})$, the *category of chain complexes*. The objects are chain complexes $A^\bullet$ given below

$$A^\bullet := (\cdots \to A^n \overset{d_A^n}{\to} A^{n+1} \overset{d_A^{n+1}}{\to} \cdots)$$

with $A^n \in \mathcal{A}, n \in \mathbb{Z}$ and where $d_A = d_A^n : A^n \to A^{n+1}$ (called the *differentials*) are morphisms in $\mathcal{A}$ such that each composite $d_A^{n+1} \circ d_A^n : A^n \to A^{n+2}$ is zero. Note that we have seen an example of a chain complex of abelian groups in Section 1.3.2.

Definition 1.4.35. A *morphism* of complexes $f : A^\bullet \to B^\bullet$ is a chain complex map, i.e. a family of morphisms $f^n : A^n \to B^n$ in $\mathcal{A}$ commuting with differentials in the sense that $f^{n+1}d_A^n = d_B^n f^n$. That is, such that the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r, "d_A^{n-2}"] & A^{n-1} \arrow[r, "d_A^{n-1}"] \arrow[d, "f^{n-1}"] & A^n \arrow[r, "d_A^n"] \arrow[d, "f^n"] & A^{n+1} \arrow[r, "d_A^{n+1}"] \arrow[d, "f^{n+1}"] & \cdots \\
\cdots \arrow[r, "d_B^{n-2}"] & B^{n-1} \arrow[r, "d_B^{n-1}"] & B^n \arrow[r, "d_B^n"] & B^{n+1} \arrow[r, "d_B^{n+1}"] & \cdots
\end{tikzcd}$$

Definition 1.4.36. For $n \in \mathbb{Z}$, the *n-th cohomology* of a complex $A^\bullet$ is

$$H^n(A^\bullet) = \ker(d_A^n)/\operatorname{im}(d_A^{n-1}).$$

If $f : A^\bullet \to B^\bullet$ is morphism of complexes, then we get induced cohomology morphisms in $\mathcal{A}$

$$H^n(f) : H^n(A^\bullet) \to H^n(B^\bullet)$$

where $H^n(f)([a^n]) := [f(a^n)]$, with $[a^n] \in \ker(d_A^n)/\operatorname{im}(d_A^{n-1})$. A morphism of complexes is a *quasi-isomorphism* if the morphisms $H^n(f) : H^n(A^\bullet) \to H^n(B^\bullet)$ are all isomorphisms.

Remark 1.4.37. The Freyd-Mitchell's embedding Theorem ([F] Theorem 4.4 and Theorem 7.34) tells us that a small abelian category (a category is called *small* if its class of objects is a set) is equivalent to a full subcategory of the category of modules over a ring, R. Since abelian categories are abstractly defined and the objects don't have elements, in general, this theorem allows us to use elements in the proofs needed in this section. However, the proofs could also be carried out using universal properties.

Definition 1.4.38. A chain complex $A^\bullet$ is called *bounded* if almost all the A^n are zero, i.e. if there exists $a, b \in \mathbb{Z}$ such that for any $n \in \mathbb{Z}$, $A^n = 0$ unless $a \leq n \leq b$. The category of bounded chain complexes is denoted $\mathcal{C}^b(\mathcal{A})$.

Theorem 1.4.39. *([W] Theorem 1.3.1) Each short exact sequence of complexes*

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

gives rise to a long exact cohomology sequence

$$\cdots \to H^{-1}(C^\bullet) \to H^0(A^\bullet) \to H^0(B^\bullet) \to H^0(C^\bullet) \to H^1(A^\bullet) \to \cdots$$

The following lemma will be useful in proofs later on. It is a well-known lemma and can be found in [W] Exercise 1.3.3.

Lemma 1.4.40. *(5-Lemma) In any commutative diagram*

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "a"] & B' \arrow[r] \arrow[d, "b"] & C' \arrow[r] \arrow[d, "c"] & D' \arrow[r] \arrow[d, "d"] & E' \arrow[d, "e"] \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E
\end{tikzcd}$$

with exact rows in any abelian category, if a, b, d and e are isomorphisms, then c is also an isomorphism.

Step 2 :

The second step in defining the derived category of an abelian category, is to define the homotopy category $\mathcal{K}(\mathcal{A})$. First we need the following definition:

Definition 1.4.41. Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes in $\mathcal{A}$ and let $h^n : A^n \to B^{n-1}$ be a collection of morphisms in $\mathcal{A}$. The morphism f is *null-homotopic* if $f^n = d_B^{n-1}h^n + h^{n+1}d_A^n$ for all $n \in \mathbb{Z}$. This can be visualised by the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & A^{n-1} \arrow[r] & A^n \arrow[r, "d_A^n"] \arrow[dl, "h^n"'] \arrow[d, "f^n"'] & A^{n+1} \arrow[r, "d_A^{n+1}"] \arrow[dl, "h^{n+1}"'] \arrow[d, "f^{n+1}"'] & A^{n+2} \arrow[r] \arrow[dl, "h^{n+2}"'] & \cdots \\
\cdots \arrow[r] & B^{n-1} \arrow[r, "d_B^{n-1}"'] & B^n \arrow[r, "d_B^n"'] & B^{n+1} \arrow[r] & B^{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

Definition 1.4.42. Let $f, g : A^\bullet \to B^\bullet$ be two morphisms of complexes in $\mathcal{A}$. We say that f and g are *homotopic* (denoted $f \sim g$) if their difference $f - g$ is null-homotopic.

Lemma 1.4.43. *Let $f, g : A^\bullet \to B^\bullet$ be two morphisms of complexes in $\mathcal{A}$. If f and g are homotopic, they induce the same morphisms $H^n(A^\bullet) \to H^n(B^\bullet)$.*

Proof. ([W] Lemma 1.4.5) By Definition 1.4.42, we know that $f - g : A^\bullet \to B^\bullet$ is null homotopic, i.e. $f^n - g^n = d_B^{n-1}h^n + h^{n+1}d_A^n$, for some collection of morphisms $h^n : A^n \to B^{n-1}$ in $\mathcal{A}$ and for all $n \in \mathbb{Z}$. Recall that $H^n(f) : \ker(d_A^n)/\operatorname{im}(d_A^{n-1}) \to \ker(d_B^n)/\operatorname{im}(d_B^{n-1})$ is given by $H^n(f)([a^n]) = [f(a^n)]$, with $[a^n] \in \ker(d_A^n)/\operatorname{im}(d_A^{n-1})$. Now let us consider $H^n(f) - H^n(g)$. We know

$$\begin{aligned}(H^n(f) - H^n(g))([a^n]) &= [f(a^n) - g(a^n)] \\ &= [d_B^{n-1}h^n(a^n) + h^{n+1}d_A^n(a^n)]\end{aligned}$$

Now the first summand, i.e. $d_B^{n-1}h^n(a^n) \in \operatorname{im} d_B^{n-1}$ and since $a^n \in \ker(d_A^n)$, the second summand is zero. Hence we get

$$H^n(f) - H^n(g) = 0.$$

In other words, f and g induce the same morphisms $H^n(A^\bullet) \to H^n(B^\bullet)$. $\square$

Definition 1.4.44. The *homotopy category* $\mathcal{K}(\mathcal{A})$ has the same objects as $\mathcal{C}(\mathcal{A})$. Its morphisms from $A^\bullet \to B^\bullet$ are the classes of morphisms of complexes $f : A^\bullet \to B^\bullet$ modulo the null-homotopic morphisms. The homotopy category in which all the complexes are bounded is denoted $\mathcal{K}^b(\mathcal{A})$.

Definition 1.4.45. A morphism $s : A^\bullet \to B^\bullet$ of $\mathcal{K}(\mathcal{A})$ is defined to be a *quasi-isomorphism* if the induced morphisms $H^n(s) : H^n(A^\bullet) \to H^n(B^\bullet)$ are invertible for all $n \in \mathbb{Z}$. We denote by Σ the class of all quasi-isomorphisms.

Step 3 :

Our aim now is to define the derived category $\mathcal{D}(\mathcal{A})$ as the 'localization' $\Sigma^{-1}\mathcal{K}(\mathcal{A})$ of the category $\mathcal{K}(\mathcal{A})$ at the class Σ. By construction $\mathcal{K}(\mathcal{A})$ is an additive category. We have the following lemma:

Lemma 1.4.46. *(a) Identities are quasi-isomorphisms and compositions of quasi-isomorphisms are quasi-isomorphisms.*

(b) Each diagram

$$A'^\bullet \xleftarrow{s} A^\bullet \xrightarrow{f} B^\bullet \quad (\textit{resp. } A'^\bullet \xrightarrow{f'} B'^\bullet \xleftarrow{s} B^\bullet)$$

of $\mathcal{K}(\mathcal{A})$ where s (resp. s') is a quasi-isomorphism, may be embedded into a square

$$\begin{tikzcd} A^\bullet \arrow[r, "f"] \arrow[d, "s"] & B^\bullet \arrow[d, "s'"] \\ A'^\bullet \arrow[r, "f'"'] & B'^\bullet \end{tikzcd}$$

which commutes in $\mathcal{K}(\mathcal{A})$.

(c) Let f be a morphism in $\mathcal{K}(\mathcal{A})$. Then there is a quasi-isomorphism s such that $sf = 0$ in $\mathcal{K}(\mathcal{A})$ if and only if there is a quasi-isomorphism t such that $ft = 0$ in $\mathcal{K}(\mathcal{A})$.

Proof. For a proof see [KS], 1.6.7. □

Clearly condition (a) of the above lemma would also be true for the preimage of Σ in the category of complexes. However, for (b) and (c) to be true, it is essential to pass to the homotopy category. Historically, this was the main reason for inserting the homotopy category between the category of complexes and the derived category. We are now ready to define the derived category, $\mathcal{D}(\mathcal{A})$.

Definition 1.4.47. The objects of $\mathcal{D}(\mathcal{A})$ are the same as the objects of $\mathcal{K}(\mathcal{A})$ and the morphisms in $\mathcal{D}(\mathcal{A})$ from $A^\bullet$ to $B^\bullet$ are given by "left-fractions", "$s^{-1} \circ f$", i.e. equivalence classes of diagrams (also called "roofs")

$$\begin{tikzcd} & B'^\bullet & \\ A^\bullet \arrow[ur, "f"] & & B^\bullet \arrow[ul, "s"'] \end{tikzcd}$$

where s is a quasi-isomorphism and a pair (f, s) is equivalent to (f', s') if and only if there is a commutative diagram of $\mathcal{K}(\mathcal{A})$

$$\begin{tikzcd}
 & B'^{\bullet} \arrow[d] & \\
A^{\bullet} \arrow[ur, "f"] \arrow[r, "f''"] \arrow[dr, "f'"'] & B'''^{\bullet} & B^{\bullet} \arrow[ul, "s"'] \arrow[l, "s''"'] \arrow[dl, "s'"] \\
 & B''^{\bullet} \arrow[u] &
\end{tikzcd}$$

where s'' is a quasi-isomorphism. The composition of (f, s) and (g, t) is defined by

$$t^{-1}g \circ s^{-1}f = (s't)^{-1} \circ g'f$$

where $s' \in \Sigma$ and g' are constructed using Lemma 1.4.46(b) from above as in the following commutative diagram of $\mathcal{K}(\mathcal{A})$

$$\begin{tikzcd}
 & & C'''^{\bullet} & & \\
 & B'^{\bullet} \arrow[ur, "g'"] & & C'^{\bullet} \arrow[ul, "s'"'] & \\
A^{\bullet} \arrow[ur, "f"] & & B^{\bullet} \arrow[ul, "s"'] \arrow[ur, "g"] & & C^{\bullet} \arrow[ul, "t"']
\end{tikzcd}$$

One can then check that composition is associative and admits identities.

Remark 1.4.48. Using 'right fractions' instead of left fractions we can obtain an isomorphic category (using Lemma 1.4.46).

The full subcategory of $\mathcal{D}(\mathcal{A})$ whose objects are bounded complexes is denoted $\mathcal{D}^b(\mathcal{A})$ and called the bounded derived category of the abelian category, $\mathcal{A}$.

Remark 1.4.49. The abelian category $\mathcal{A}$ becomes a full subcategory of $\mathcal{D}^b(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ by sending an object A to the complex which has A at position 0 and 0 elsewhere.

Triangles as generalized short exact sequences

In this section we introduce some diagrams in derived categories, called exact triangles. These may be thought of as analogues of short exact sequences in the sense that they both give rise to long exact cohomology sequences (Theorem 1.4.39). First let me introduce some notation and concepts needed later. Let $\mathcal{A}$ be an abelian category. All complexes below are assumed to be objects of $\mathcal{C}^b(\mathcal{A})$.

Definition 1.4.50. Fix an integer n and for any complex $A^\bullet = (A^i, d_A^i)$, we define a new complex $A[n]^\bullet$ by $(A[n])^i = A^{n+i}, d^k_{A[n]} = (-1)^n d_A^{k+n}$. For a morphism of complexes $f : A^\bullet \to B^\bullet$, let $f[n] : A[n]^\bullet \to B[n]^\bullet$ coincide with f componentwise. We then have a functor $T^n : \mathcal{C}(\mathcal{A}) \to \mathcal{C}(\mathcal{A})$, $T^n(A^\bullet) = A[n]^\bullet, T^n(f) = f[n]$. This is called a *translation by n* functor. Note that translation shifts homology

$$H^i(A[n]^\bullet) = H^{i+n}(A^\bullet)$$

Now let us give some definitions that are needed to formulate the next lemma.

Definition 1.4.51. Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes. The *cone* of f is the following complex $\mathrm{cone}(f)$:

$$\mathrm{cone}(f)^n = A[1]^n \oplus B^n, \quad d_{\mathrm{cone}(f)}(a^{n+1}, b^n) = (-d_A a^{n+1}, f(a^{n+1}) + d_B b^n)$$

with $a^{n+1} \in A^{n+1}, b^n \in B^n$. It is useful to write the differential as a matrix, so that

$$d^n_{\mathrm{cone}(f)} = \begin{pmatrix} -d_A^{n+1} & 0 \\ f^{n+1} & d_B^n \end{pmatrix}.$$

Note that $\mathrm{cone}(f)$ sits in a short exact sequence

$$0 \to B^\bullet \to \mathrm{cone}(f) \to A[1]^\bullet \to 0$$

of chain complexes, where the left map sends b to $(0, b)$, and the right map sends (a, b) to (a).

Definition 1.4.52. The *cylinder* $\mathrm{cyl}(f)$ of a morphism $f : A^\bullet \to B^\bullet$ is the following complex:

$$\mathrm{cyl}(f) = A^\bullet \oplus A[1]^\bullet \oplus B^\bullet$$

$$d^n_{\mathrm{cyl}(f)}(a^n, a^{n+1}, b^n) = (d_A a^n - a^{n+1}, -d_A a^{n+1}, f(a^{n+1}) + d_B b^n)$$

with $a^n \in A^n, a^{n+1} \in A^{n+1}, b^n \in B^n$. The differential is given by the following matrix

$$d^n_{\mathrm{cyl}(f)} = \begin{pmatrix} d^n_A & -\mathrm{id}_A & 0 \\ 0 & -d^{n+1}_A & 0 \\ 0 & f^{n+1} & d^n_B \end{pmatrix}.$$

Now in order to introduce distinguished triangles and to prove the next proposition we need the following lemma.

Lemma 1.4.53. *For any morphism $f : A^\bullet \to B^\bullet$ of complexes there exists the following commutative diagram in $\mathcal{C}^b(\mathcal{A})$ with exact rows:*

$$\begin{tikzcd}
 & 0 \arrow[r] & B^\bullet \arrow[r, "\overline{\pi}"] \arrow[d, "\alpha"] & \mathrm{cone}(f) \arrow[r, "\delta=\delta(f)"] \arrow[d, equal] & A[1]^\bullet \arrow[r] & 0 \\
0 \arrow[r] & A^\bullet \arrow[r, "\overline{f}"] \arrow[d, equal] & \mathrm{cyl}(f) \arrow[r, "\pi"] \arrow[d, "\beta"] & \mathrm{cone}(f) \arrow[r] & 0 & \\
 & A^\bullet \arrow[r, "f"] & B^\bullet & & &
\end{tikzcd} \tag{1.3}$$

Moreover $\beta \circ \alpha = \mathrm{id}_B$ and $\alpha \circ \beta$ is homotopic to $\mathrm{id}_{\mathrm{cyl}(f)}$. In particular, α and β are quasi-isomorphisms.

Proof. ([GM], III.3) The definitions of the morphisms in $\mathcal{C}^b(\mathcal{A})$ are as follows:

$$\pi^i : \mathrm{cyl}(f)^i = A^i \oplus A^{i+1} \oplus B^i \to \mathrm{cone}(f)^i = A^{i+1} \oplus B^i, (a^i, a^{i+1}, b^i) \mapsto (a^{i+1}, b^i)$$

$$\overline{\pi}^n : B^n \to \mathrm{cone}(f)^n = A^{n+1} \oplus B^n, \quad b^n \mapsto (0, b^n).$$

$$\overline{f}^n : A^n \to \mathrm{cyl}(f)^n = A^n \oplus A^{n+1} \oplus B^n, \quad a^n \mapsto (a^n, 0, 0).$$

$$\alpha^n : B^n \to \mathrm{cyl}(f)^n = A^n \oplus A^{n+1} \oplus B^n, \quad b^n \mapsto (0, 0, b^n).$$

$$\beta^n : \mathrm{cyl}(f)^n = A^n \oplus A^{n+1} \oplus B^n \to B^n, \quad (a^n, a^{n+1}, b^n) \mapsto f(a^n) + b^n.$$

$$\delta^n : \mathrm{cone}(f)^n = A^{n+1} \oplus B^n \to A^{n+1}, \quad (a^{n+1}, b^n) \mapsto a^{n+1}.$$

The fact that these really are morphisms in $\mathcal{C}^b(\mathcal{A})$, that the diagram is commutative and that the rows are exact is verified in [GM], III.3. From these definitions $\beta \circ \alpha = \mathrm{id}_B$ is obvious. It remains to show then that $\alpha \circ \beta$ is homotopic to $\mathrm{id}_{\mathrm{cyl}(f)}$. Define h^i as follows:

$$h^n : \mathrm{cyl}(f)^n = A^n \oplus A^{n+1} \oplus B^n \to \mathrm{cyl}(f)^{n-1} = A^{n-1} \oplus A^n \oplus B^n,$$

$$(a^n, a^{n+1}, b^n) \mapsto (0, a^n, 0).$$

So from this we have

$$d_{\mathrm{cyl}(f)} h^n(a^n, a^{n+1}, b^n) = (-a^n, -d_A a^n, f(a^n))$$

and

$$h^n d_{\mathrm{cyl}(f)}(a^n, a^{n+1}, b^n) = (0, d_A a^n - a^{n+1}, 0).$$

Hence we get

$$(h^n d^{n-1}_{\mathrm{cyl}(f)} + d^{n+1}_{\mathrm{cyl}(f)} h^n)(a^n, a^{n+1}, b^n) = (-a^n, -a^{n+1}, f(a^n))$$

On the other hand

$$(\alpha \circ \beta - \mathrm{id}_{\mathrm{cyl}(f)})(a^n, a^{n+1}, b^n) = (-a^n, -a^{n+1}, f(a^n))$$

Hence $\alpha \circ \beta - \mathrm{id}_{\mathrm{cyl}(f)} = hd + dh$, i.e. α and β are homotopic inverses of each other. $\square$

Definition 1.4.54. a) A *triangle* in a category whose objects are complexes is a diagram of the form

$$A^\bullet \xrightarrow{u} B^\bullet \xrightarrow{v} C^\bullet \xrightarrow{w} A[1]^\bullet,$$

where u, v, w are morphisms in that category.

b) A *morphism of triangles* in such a category is a commutative diagram of the form

$$\begin{array}{ccccccc}
A^\bullet & \xrightarrow{u} & B^\bullet & \xrightarrow{v} & C^\bullet & \xrightarrow{w} & A[1]^\bullet \\
\Big\downarrow f & & \Big\downarrow g & & \Big\downarrow h & & \Big\downarrow f[1] \\
A'^\bullet & \xrightarrow{u'} & B'^\bullet & \xrightarrow{v'} & C'^\bullet & \xrightarrow{w'} & A'[1]^\bullet
\end{array}$$

Such a morphism is said to be an isomorphism if f, g, h are isomorphisms in that category.

c) A triangle in $\mathcal{K}^b(\mathcal{A})$ or $\mathcal{D}^b(\mathcal{A})$ is said to be *distinguished* if it is isomorphic to the triangle

$$A^\bullet \xrightarrow{\overline{f}} \mathrm{cyl}(f) \xrightarrow{\pi} \mathrm{cone}(f) \xrightarrow{\delta} A[1]^\bullet$$

from Lemma 1.4.53 for some morphism of complexes $f : A^\bullet \to B^\bullet$.

Remark 1.4.55. Lemma 1.4.53 shows that for any morphism of complexes $f : A^\bullet \to B^\bullet$, the triangle

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\overline{\pi}} \mathrm{cone}(f) \xrightarrow{\delta} A[1]^\bullet$$

is distinguished in $\mathcal{K}^b(\mathcal{A})$ and $\mathcal{D}^b(\mathcal{A})$.

The next proposition shows that any short exact sequence of complexes can be completed to a distinguished triangle.

Proposition 1.4.56. *If*

$$0 \longrightarrow A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \longrightarrow 0 \tag{1.4}$$

is a short exact sequence of complexes, then there exists a distinguished triangle

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{\overline{\delta}} A[1]^\bullet$$

in $\mathcal{D}^b(\mathcal{A})$.

Proof. Using notation from Lemma 1.4.53, we have a commutative diagram in $\mathcal{C}^b(\mathcal{A})$ with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "g"] & C^\bullet \arrow[r] & 0 \\ 0 \arrow[r] & A^\bullet \arrow[r, "\overline{f}"] \arrow[u, equal] & \mathrm{cyl}(f) \arrow[r, "\pi"] \arrow[u, "\beta"'] & \mathrm{cone}(f) \arrow[r] \arrow[u, "\gamma"'] & 0 \end{tikzcd} \tag{1.5}$$

where $\gamma^n : \mathrm{cone}(f)^n \to C^n$ is given by $\gamma^n(a^{n+1}, b^n) := g^n(b^n)$. Let us verify first that γ is a morphism of complexes.

$$\begin{aligned} \gamma^{n+1} d^n_{\mathrm{cone}(f)}(a^{n+1}, b^n) &= \gamma^{n+1}(-d_A^{n+1}a^{n+1}, f^{n+1}(a^{n+1}) + d_B^n b^n) \\ &= g^{n+1}(f^{n+1}(a^{n+1}) + d_B^n b^n). \end{aligned}$$

Since (1.4) above is a short exact sequence, we know $g^{n+1}(f^{n+1}(a^{n+1})) = 0$. Hence $\gamma^{n+1}d^n_{\mathrm{cone}(f)}(a^{n+1}, b^n) = g^{n+1}(f^{n+1}(a^{n+1}) + d_B^n b^n) = g^{n+1}(d_B^n b^n)$. Now $d_C^n(\gamma^n(a^{n+1}, b^n)) = d_C^n(g^n(b^n))$. But g is a morphism of complexes, so we get $d_C^n(\gamma^n(a^{n+1}, b^n)) = g^{n+1}(d_B^n, b^n)$. Hence γ is a morphism of complexes.

Now let us verify that the right square in (1.5) is commutative. Firstly $g^n(\beta^n(a^n, a^{n+1}, b^n)) = g^n(f(a^n) + b^n) = g^n(b^n)$. Also, $\gamma^n(\pi(a^n, a^{n+1}, b^n)) = \gamma(0, b^n) = g^n(b^n)$. Hence the right square is commutative.

We know β is a quasi-isomorphism, hence the 5-lemma (Lemma 1.4.40) implies that γ is a quasi-isomorphism aswell. Hence the roof

$$\begin{tikzcd} & \mathrm{cone}(f) \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\ C^\bullet & & A[1]^\bullet \end{tikzcd}$$

where δ (as defined in Lemma 1.4.53) gives a morphism $\overline{\delta} : C^\bullet \to A[1]^\bullet$ in $\mathcal{D}^b(\mathcal{A})$ such that the diagram

$$\begin{tikzcd} A^\bullet \arrow[r, "f"] \arrow[d, equal] & B^\bullet \arrow[r, "\overline{\pi}"] \arrow[d, equal] & \mathrm{cone}(f) \arrow[r, "\delta"] \arrow[d, "\gamma"] & A[1]^\bullet \arrow[d, equal] \\ A^\bullet \arrow[r, "f"] & B^\bullet \arrow[r, "g"] & C^\bullet \arrow[r, "\overline{\delta}"] & A[1]^\bullet \end{tikzcd} \tag{1.6}$$

(with $\overline{\pi}$ also as in Lemma 1.4.53) gives the required isomorphism in $\mathcal{D}^b(\mathcal{A})$. $\square$

Remark 1.4.57. There is also the notion of a triangulated category, which generalises the structure that distinguished triangles give to $\mathcal{K}(\mathcal{A})$. There are a number of axioms to be satisfied in order for a category to be a triangulated category (See [GM] Section IV.1, [W] Section 10.2). We do not need this level of generality for this thesis, except to note that the derived category of an abelian category is in fact also a triangulated category.

1.4.4 Derived functors

We begin with a proposition to serve as a motivation for the notion of derived functors.

Proposition 1.4.58. *Let $\mathcal{A}$ be an abelian category. Then $F = \mathrm{Hom}_{\mathcal{A}}(N, -)$ is a left exact functor from $\mathcal{A}$ to* **Ab** *for every object N in $\mathcal{A}$. That is, according to Definition 1.4.22, given a short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{1.7}$$

in $\mathcal{A}$, we get the following exact sequence of abelian groups:

$$0 \longrightarrow \mathrm{Hom}(N, A) \xrightarrow{F(f)} \mathrm{Hom}(N, B) \xrightarrow{F(g)} \mathrm{Hom}(N, C)\ . \tag{1.8}$$

Proof. ([W] Proposition 1.6.8) If $\alpha \in \mathrm{Hom}(N, A)$ then $F(f)(\alpha) = f \circ \alpha$. We know since (1.7) is exact, that f is a monomorphism. Hence if $f \circ \alpha = F(f)(\alpha) = 0$, then α must be zero. This implies that $F(f)$ is a monomorphism.

Again since (1.7) is exact, we know $g \circ f = 0$. This gives us that $F(g)F(f)(\alpha) = g \circ f \circ \alpha = 0$, so $F(g)F(f) = 0$. It remains to show that $\mathrm{im}(F(f)) \supseteq \ker(F(g))$, i.e. if $\beta \in \mathrm{Hom}(N, B)$ is such that $F(g)(\beta) = g \circ \beta$ is zero, then $\beta = f \circ \alpha$ for some α. Exactness of (1.7) implies that $f : A \to B$ is the kernel of g. The universal property of $\ker(g)$ gives then that β factors through $f : A \to B$ if $g \circ \beta = 0$. $\square$

Now if we want to extend the sequence (1.8) to the right, we need the notion of a right derived functor. In order to define this we need the following definition.

Definition 1.4.59. An object M in an abelian category $\mathcal{A}$ is called *injective* if it satisfies the following universal lifting property: Given an monomorphism $f : A \to B$ and a morphism $\alpha : A \to M$, there exists at least one morphism $\beta : B \to M$ such that $\alpha = \beta \circ f$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[dl, dashed, "\beta"] \\ & M & \end{tikzcd}$$

We say that $\mathcal{A}$ has *enough injectives* if for every object A in $\mathcal{A}$ there is a monomorphism $A \to M$ with M injective.

Lemma 1.4.60. *An object M in an abelian category $\mathcal{A}$ is injective if and only if $F = \mathrm{Hom}_{\mathcal{A}}(-, M)$ is an exact functor. That is, if and only if the sequence of groups*

$$0 \longrightarrow \mathrm{Hom}(C, M) \xrightarrow{F(g)} \mathrm{Hom}(B, M) \xrightarrow{F(f)} \mathrm{Hom}(A, M) \longrightarrow 0$$

is exact for every exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *in* $\mathcal{A}$.

Proof. ([W] Exercise 2.5.1) Suppose that $F = \mathrm{Hom}(-, M)$ is exact and that we are given a monomoprhism $f : A \to B$ and a morphism $\alpha : A \to M$. We can lift $\alpha \in \mathrm{Hom}(A, M)$ to $\beta \in \mathrm{Hom}(B, M)$ such that $\alpha = F(f)(\beta) = \beta \circ f$ because $F(f)$ is an epimorphism. Thus M is injective as it has the universal lifting property as in Definition 1.4.59.

Conversely, suppose that M in injective. To show that F is exact, it suffices to show that $F(f)$ is a monomorphism for every short exact sequence as above. Given $\alpha \in \mathrm{Hom}(A, M)$, the universal lifting property of M gives $\beta \in \mathrm{Hom}(B, M)$ so that $\alpha = \beta \circ f = F(f)(\beta)$, i.e. $F(f)$ is a monomorphism. $\square$

Definition 1.4.61. Let M be an object of $\mathcal{A}$, an abelian category. An *injective resolution* is a complex $I^\bullet$ of injective objects, with $I^i = 0$ for $i < 0$ and a morphism $M \to I^0$ such that the complex

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$

is exact.

Lemma 1.4.62. *([W] Lemma 2.3.6) If an abelian category $\mathcal{A}$ has enough injectives, then every object in $\mathcal{A}$ has an injective resolution.*

Definition 1.4.63. If $F : \mathcal{A} \to \mathcal{B}$ is a left exact functor and $\mathcal{A}$ has enough injectives, the *right derived functors* R^iF can be defined as follows:

$$R^iF(M) := H^i(F(I^\bullet))$$

where $I^\bullet$ is an injective resolution of M.

Remark 1.4.64. Applying a left exact functor F to an exact sequence

$$0 \to M \to I^0 \to I^1$$

gives

$$0 \to F(M) \to F(I^0) \to F(I^1)$$

hence $F(M) \cong R^0F(M)$. If M itself is injective, we can use $I^0 = M, I^i = 0$ for all $i > 0$ and obtain $R^iF(M) = 0$ for all $i > 0$.

Example 1.4.65. The cohomology groups as defined in Section 1.3.2 are an example of a right derived functor of sheaves of abelian groups on a complex manifold X, i.e. $H^i(X, \mathcal{F}) = R^i\Gamma(\mathcal{F})$, where $\mathcal{F}$ is a sheaf of abelian groups on X.

Definition 1.4.66. Given an object M with an injective resolution

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$$

we define the Ext functor, i.e. the right derived functor of $\mathrm{Hom}(N, -)$ as follows: $\mathrm{Ext}^i(N, M) := H^i(\mathrm{Hom}(N, I^\bullet))$. In particular, because of the left exactness of $\mathrm{Hom}(N, -)$ we have $\mathrm{Ext}^0(N, M) = \mathrm{Hom}(N, M)$.

Theorem 1.4.67. *([H] Theorem III.1.1A) Let F be an additive functor. Each short exact sequence*

$$0 \to M' \to M \to M'' \to 0$$

of objects of $\mathcal{A}$, gives rise to a long exact sequence

$$0 \to R^0F(M') \to R^0F(M) \to R^0F(M'') \to R^1F(M') \to R^1F(M) \to \cdots$$

As an example consider a short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of objects of $\mathcal{A}$. Applying the Ext functor to this sequence we get a long exact sequence

$$0 \to \mathrm{Hom}(N, M') \to \mathrm{Hom}(N, M) \to \mathrm{Hom}(N, M'') \to \mathrm{Ext}^1(N, M') \to \cdots$$

If M' is injective, then from Remark 1.4.64 we get $\mathrm{Ext}^1(N, M') = 0$, hence the Hom sequence

$$0 \to \mathrm{Hom}(N, M') \to \mathrm{Hom}(N, M) \to \mathrm{Hom}(N, M'') \to 0.$$

is exact.

Remark 1.4.68. If $\mathcal{A}$ is an abelian category with enough injectives

$$\mathrm{Ext}^i(A, B) \cong \mathrm{Hom}_{\mathcal{D}^b(\mathcal{A})}(A^\bullet, B[i]^\bullet)$$

for all $A, B \in \mathcal{A}$ ([W] Section 10.7).

Chapter 2

Stability of vector bundles

2.1 Vector bundles

We now want to introduce the notion of a holomorphic vector bundle. A line bundle (Definition 1.2.6) is in particular a vector bundle of rank 1. Again, throughout this chapter X will denote a complex manifold, unless otherwise specified.

Definition 2.1.1. A *holomorphic vector bundle* of rank r is a holomorphic map $p : E \to X$ of complex manifolds which satisfies the following conditions:

1. For any point $x \in X$, the preimage $E_x := p^{-1}(x)$ (called a *fibre*) has a structure of an r-dimensional $\mathbb{C}$-vector space.

2. The mapping p is locally trivial, i.e. for any point $x \in X$, there exists an open neighbourhood U_i containing x and a biholomorphic map $\varphi_i : p^{-1}(U_i) \to U_i \times \mathbb{C}^r$ such that the diagram

$$\begin{tikzcd} p^{-1}(U_i) \arrow[rr, "\varphi_i"] \arrow[dr, "p"'] & & U_i \times \mathbb{C}^r \arrow[dl, "\mathrm{pr}_1"] \\ & U_i & \end{tikzcd}$$

commutes.

Moreover, φ_i takes the vector space E_x isomorphically onto $\{x\} \times \mathbb{C}^r$ for each $x \in U_i$; φ_i is called a *trivialisation* of E over U_i. Note that for any pair of trivialisations φ_i and φ_j the map

$$g_{ij} : U_i \cap U_j \to \mathrm{GL}(r, \mathbb{C})$$

given by

$$g_{ij}(x) = \varphi_i \circ (\varphi_j)^{-1}|_{\{x\}\times\mathbb{C}^r}, \text{ i.e. } \varphi_i(\varphi_j^{-1}(x, v)) = (x, g_{ij}(x)v)$$

is holomorphic; the maps g_{ij} are called *transition functions* for E relative to the trivialisations φ_i, φ_j. The transition functions of E necessarily satisfy the identities

$$g_{ij}(x) \circ g_{ji}(x) = \mathrm{id} \qquad \text{for all } x \in U_i \cap U_j$$

$$g_{ij}(x) \circ g_{jk}(x) \circ g_{ki}(x) = \mathrm{id} \qquad \text{for all } x \in U_i \cap U_j \cap U_k.$$

Conversely, given an open cover $\{U_i\}$ of X and transition functions $g_{ij} : U_i \cap U_j \to \mathrm{GL}(r, \mathbb{C})$, for all i, j satisfying the identities above, then we can define a vector bundle, E with transition functions g_{ij} using the glueing construction as follows: We glue $U_i \times \mathbb{C}^r$ together by taking the union over all i of $U_i \times \mathbb{C}^r$ to get $E := \bigsqcup (U_i \times \mathbb{C}^r)/\sim$, where $(x, v) \sim (x, g_{ij}(x)(v))$, for all $x \in U_i \cap U_j, v \in \mathbb{C}^r$.

Example 2.1.2. The simplest example is known as the *trivial vector bundle* of rank r, i.e. $\mathrm{pr}_1 : X \times \mathbb{C}^r \to X$.

Example 2.1.3. Let $\mathbb{P}^n$ be the complex projective space as described in Example 1.1.6, i.e. $\mathbb{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$ where $(z_0, \ldots, z_n) \sim (\lambda z_0, \ldots, \lambda z_n)$ for all $\lambda \in \mathbb{C}^*$. This means each line $\ell \subset \mathbb{C}^{n+1}$ through the origin corresponds to a point $[\ell] \in \mathbb{P}^n$. The set

$$\mathcal{O}_{\mathbb{P}^n}(-1) = \{([\ell], z) \in \mathbb{P}^n \times \mathbb{C}^{n+1} | z \in \ell\}$$

forms in a natural way a line bundle over $\mathbb{P}^n$. To see this, consider the projection $\mathrm{pr}_1 : \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathbb{P}^n$, i.e. the projection to the first factor. Let $\mathbb{P}^n = \bigcup_{i=0}^n U_i$ be the standard open cover as described in Example 1.1.6. A trivialisation of $\mathcal{O}_{\mathbb{P}^n}(-1)$ over U_i is given by $\varphi_i : p^{-1}(U_i) \to U_i \times \mathbb{C}, (\ell, z) \mapsto (\ell, z_i)$. The transition maps $g_{ij}(\ell) : \mathbb{C} \to \mathbb{C}$ are given by $w \mapsto \frac{z_i}{z_j} \cdot w$, where $\ell = (z_0 : \cdots : z_n)$.

Definition 2.1.4. Let F and E be vector bundles of rank r and n respectively, with $r \leq n$ and $F \subset E$ is a submanifold. Then, F is called a *subbundle* of E if there exists an open cover $\{U_i\}$ and transition functions $g_{ij} : U_i \cap U_j \to \mathrm{GL}(r, \mathbb{C})$ for F and $h_{ij} : U_i \cap U_j \to \mathrm{GL}(n, \mathbb{C})$ for E such that

$$h_{ij}(x) = \begin{pmatrix} g_{ij}(x) & * \\ 0 & k_{ij}(x) \end{pmatrix}.$$

The *quotient bundle* $G = E/F$ is described by transition functions k_{ij}.

Definition 2.1.5. Let E be a vector bundle on X (with open over $\{U_i\}$) and let g_{ij} be transition functions of E. The *dual bundle*, E^*, of E is given by transition functions

$$h_{ij}(x) := {}^t g_{ij}(x)^{-1} \quad \forall x \in U_i \cap U_j$$

Definition 2.1.6. Let $p : E \to X$ and $p' : E' \to X$ be two complex vector bundles on X. A holomorphic map $f : E \to E'$ is called a *morphism* of vector bundles if the diagram

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

commutes and for each point $x \in X$ the map $f|_{E_x} : E_x \to E'_x$ is a homomorphism of vector spaces.

Example 2.1.7. Consider the line bundles $p : \mathcal{O}(-1) \to \mathbb{P}^1$ and $p' : \mathcal{O} \to \mathbb{P}^1$ on $\mathbb{P}^1$ with coordinates $(z_0 : z_1)$ and standard open cover $\{U_0, U_1\}$. Recall from Example 2.1.3 that $\mathcal{O}(-1) = \{([\ell], v) \in \mathbb{P}^1 \times \mathbb{C}^2 | v \in \ell\}$. We can also write this as

$$\mathcal{O}(-1) = \big\{((z_0 : z_1), (v_0, v_1)) | (v_0, v_1) \in \mathbb{C} \cdot (z_0, z_1)\big\}.$$

Recall also that $\mathcal{O} = \{((z_0 : z_1), v) \in \mathbb{P}^1 \times \mathbb{C}\}$.

Let $f : \mathcal{O}(-1) \to \mathcal{O}$ be the morphism of vector bundles, given by

$$f((z_0 : z_1), (v_0, v_1)) = ((z_0 : z_1), v_0)$$

Clearly

$$\begin{tikzcd} \mathcal{O}(-1) \arrow[rr, "f"] \arrow[dr, "p"'] & & \mathcal{O} \arrow[dl, "p'"] \\ & \mathbb{P}^1 & \end{tikzcd}$$

commutes. If $(z_0 : z_1) \neq (0 : 1) \in \mathbb{P}^1$,

$$\mathcal{O}(-1)_{(z_0:z_1)} \cong \{(z_0 : z_1)\} \times \mathbb{C}.$$

via $f|_{\mathcal{O}(-1)_{(z_0:z_1)}}$. If $(z_0 : z_1) = (0 : 1)$,

$$\mathcal{O}(-1)_{(0:1)} = \{((0 : 1), (0, v_1)) | v_1 \in \mathbb{C}\}$$

and $f|_{\mathcal{O}(-1)_{(0:1)}} = 0$.

Hence for all $(z_0 : z_1) \in \mathbb{P}^1$, the map

$$f|_{\mathcal{O}(-1)_{(z_0:z_1)}} : \mathcal{O}(-1)_{(z_0:z_1)} \to \mathcal{O}_{(z_0:z_1)}$$

is a homomorphism of vector spaces. This verifies that $f : \mathcal{O}(-1) \to \mathcal{O}$ really is a morphism of vector bundles.

Let E and E' be vector bundles over X with rank r and r', respectively and let $\{U_i\}$ be an open cover of X such that E and E' are trivial over U_i for each i. A morphism $f : E \to E'$ can be described locally by holomorphic functions, f_i, as follows. For each i, using trivialisations of E and E', f induces maps

$$U_i \times \mathbb{C}^r \to U_i \times \mathbb{C}^{r'}, \quad (x, v) \mapsto (x, f_i(x)v)$$

where $f_i : U_i \to \mathrm{Mat}_{r' \times r}(\mathbb{C})$. These holomorphic functions necessarily satisfy

$$f_i(x) \circ g_{ij}(x) = g'_{ij}(x) \circ f_j(x) \text{ for all } x \in U_i \cap U_j \tag{2.1}$$

where g_{ij} and g'_{ij} are transition functions of E and E', respectively. Note that a set of functions $\{f_i\}$ defines an isomorphism of vector bundles if an only if $f_i(x)$ are invertible matrices for all i and x.

Remark 2.1.8. Let $f : L \to L'$ be line bundles on $\mathbb{P}^1$, with open cover $\{U_0, U_1\}$ and let g_{ij} and g'_{ij} be transition functions for L and L', respectively. The above definition implies that L and L' are isomorphic if $g'_{01} = g_{01} \cdot \frac{f_0}{f_1}$ on $U_0 \cap U_1$, where $f_0 : U_0 \cong \mathbb{C} \to \mathbb{C}^*$ and $f_1 : U_1 \cong \mathbb{C} \to \mathbb{C}^*$ are arbitrary. For example, the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$ given by transition functions $g_{01} = \frac{z_1^n}{z_0^n}$ is isomorphic to the line bundle given by transition functions $g'_{01} = -\frac{z_1^n}{z_0^n}$.

Definition 2.1.9. Let E be a vector bundle of rank r and L a line bundle on X and let $\{U_i\}$ be an open cover of X such that E and L are trivial on each U_i. Let $g_{ij} : U_i \cap U_j \to \mathrm{GL}(r, \mathbb{C})$ and $h_{ij} : U_i \cap U_j \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$ be transition functions of E and L, respectively. We define $E \otimes L$, the *tensor product* of E and L to be the vector bundle given by transition functions $f_{ij} : U_i \cap U_j \to \mathrm{GL}(r, \mathbb{C})$, where

$$f_{ij}(x) := h_{ij}(x) \cdot g_{ij}(x) \text{ for all } x \in U_i \cap U_j.$$

Definition 2.1.10. A sequence of morphisms of vector bundles over X

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

is an exact sequence of vector bundles if

$$0 \longrightarrow E'_x \longrightarrow E_x \longrightarrow E''_x \longrightarrow 0 \tag{2.2}$$

is an exact sequence of vector spaces for all $x \in X$ (See Definition 1.4.11). The vector bundle E' is called a subbundle of E, and E'' is called a quotient bundle of E.

We also say that an exact sequence

$$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

is an extension of E'' by E'.

Remark 2.1.11. Since the dimension of vector spaces is additive on exact sequences, i.e. given a short exact sequence of vector spaces such as (2.2), we have $\dim(E_x) = \dim(E'_x) + \dim(E''_x)$. Hence, the rank of vector bundles is additive as well, i.e. $\mathrm{rk}(E') = \mathrm{rk}(E') + \mathrm{rk}(E'')$ in the notation of Definition 2.1.10.

In the same way as we defined a holomorphic section of a line bundle (Definition 1.2.15), we can also define a holomorphic section of a vector bundle.

Definition 2.1.12. Let $p : E \to X$ be a vector bundle on X and let U be an open set in X. A holomorphic map $s : U \to E$ is called a *holomorphic section* of E over U if $p \circ s = \mathrm{id}_U$. The set of all holomorphic sections is denoted $\Gamma(U, E)$ and $U \mapsto \Gamma(U, E)$ forms a sheaf called the *sheaf of sections* of the vector bundle. Sections over X are called *global sections* of E. Global sections can be added and multiplied with a scalar, so the space of global sections is in fact a vector space. It will be denoted by $H^0(X, E)$ (or $H^0(E)$ if it clear which X we are referring to) or $\Gamma(X, E)$.

Definition 2.1.13. Let $f : E' \to E$ be a morphism of vector bundles. This induces a *linear map of spaces of sections* $H^0(f) : H^0(E') \to H^0(E)$, defined by $H^0(f)(s') := f \circ s'$.

Let us now examine the correspondence between locally free sheaves (Definiton 1.3.20) and vector bundles.

Theorem 2.1.14. *Sending a holomorphic vector bundle to its sheaf of sections gives a bijection between the set of holomorphic vector bundles of rank r and the set of locally free $\mathcal{O}_X$-modules of rank r.*

Proof. ([Hu] Proposition 2.36) Recall by Definition 1.3.20 that a locally free $\mathcal{O}_X$-module of rank r is a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on X which is locally isomorphic to $\mathcal{O}_X^{\oplus r}$. Consider a vector bundle $p : E \to X$ of rank r over X. Let $\mathcal{E}$ denote the sheaf of sections of E. Because $X \times \mathbb{C}^r$ has sheaf of sections $\mathcal{O}_X^{\oplus r}$, $\mathcal{E}$ is locally free since E is locally isomorphic to $X \times \mathbb{C}^r$.

Conversely, let $\mathcal{E}$ be a locally free sheaf of rank r. If we have chosen trivialisations $\varphi_i : \mathcal{E}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus r}$, of $\mathcal{E}$ then the transition functions

$$g_{ij} := \varphi_i \circ (\varphi_j^{-1})|_{U_i \cap U_j} : \mathcal{O}_{U_i \cap U_j}^{\oplus r} \cong \mathcal{O}_{U_i \cap U_j}^{\oplus r}$$

are given by multiplication with an invertible matrix of holomorphic functions on $U_i \cap U_j$. Define the vector bundle, E, on X with an open cover $\{U_i\}$, to be the vector bundle given by transition functions g_{ij}. It is easy to see that these two constructions are inverse to each other. □

Notation : We will usually denote by $\mathcal{E}$ the sheaf corresponding to a vector bundle E.

Remark 2.1.15. Using Definition 2.1.12, it is not hard to show that we have an exact sequence in the category of coherent sheaves

$$0 \longrightarrow \mathcal{O}(-1) \xrightarrow{f} \mathcal{O} \longrightarrow \mathbb{C}_P \longrightarrow 0$$

where $f : \mathcal{O}(-1) \to \mathcal{O}$ is the morphism on $\mathbb{P}^1$ as described in Example 2.1.7 and $\mathbb{C}_P$ is the skyscraper sheaf at $P = (0 : 1)$ with stalk $\mathbb{C}$, i.e. if U is an open set of $\mathbb{P}^1$ containing P, then $\mathbb{C}_P(U) = \mathbb{C}$. If U does not contain P, then $\mathbb{C}_P(U) = 0$. In particular, f is a monomorphism in $\mathbf{Coh}(\mathbb{P}^1)$, the category of coherent sheaves on $\mathbb{P}^1$ but not an epimorphism, so it is not an isomorphism.

On the other hand, it can be shown that f is a monomorphism and an epimorphism in the category of vector bundles on $\mathbb{P}^1$, however it is not an isomorphism. This implies that f is not a strict morphism in the category of vector bundles. Examples like these serve as a motivation to extend the category of vector bundles to the category of coherent sheaves.

Remark 2.1.16. Let $f \in \mathrm{Hom}(E', E)$ be a morphism of vector bundles on a smooth curve C which is injective on generic fibres, i.e. there exists an open dense subset $U \subset C$ such that

$$f_x : E'_x \to E_x$$

is injective for all $x \in U$.

The kernel of f is a subsheaf of the locally free sheaf E', hence it is locally free (See Remark 1.3.22) But our assumption implies that $\ker f_x = 0$ for all $x \in U$, hence $\ker f = 0$ and we obtain an exact sequence

$$0 \longrightarrow E' \xrightarrow{f} E \xrightarrow{g} \mathcal{F} \longrightarrow 0$$

where $\mathcal{F}$ is a coherent sheaf on C. By modding out the torsion of $\mathcal{F}$, we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & E' \arrow[r, "f"] & E \arrow[r, "g"] \arrow[d, equal] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker g' \arrow[r, "f'"'] & E \arrow[r, "g'"'] & \mathcal{G} \arrow[r] & 0
\end{tikzcd}$$

where $\mathcal{G} := \mathcal{F}/\operatorname{tors}(\mathcal{F})$. Because $\mathcal{F}/\operatorname{tors}(\mathcal{F})$ is locally free (see Remark 1.3.22), $f' : \ker g' \to E$ is a sub*bundle* of E. The process described above is referred to as *saturation*.

2.1.1 Cohomology

Let E be a vector bundle and $\mathcal{E}$ be its sheaf of holomorphic sections (recall that these are in bijection from Theorem 2.1.14). We can define cohomology groups, $H^i(X, E) := H^i(\mathcal{E})$, using Čech cohomology described in Section 1.3.2. Therefore given a short exact sequence of vector bundles over X as follows

$$0 \longrightarrow E' \xrightarrow{f} E \xrightarrow{g} E'' \longrightarrow 0,$$

using short exact sequences of its sheaf of holomorphic sections $\mathcal{E}$, we get an induced long exact cohomology sequence as follows

$$0 \to H^0(X, E') \to H^0(X, E) \to H^0(X, E'') \to H^1(X, E') \to \cdots$$

The exactness of

$$0 \longrightarrow H^0(X, E') \xrightarrow{H^0(f)} H^0(X, E) \xrightarrow{H^0(g)} H^0(X, E'') \tag{2.3}$$

can be shown easily by proving that $H^0(f)$ is injective and $\mathrm{im}(H^0(f)) = \ker(H^0(g))$. Recall first that

$$0 \longrightarrow E' \xrightarrow{f} E \xrightarrow{g} E''$$

is exact if and only if

$$0 \longrightarrow E'_x \xrightarrow{f_x} E_x \xrightarrow{g_x} E''_x$$

is exact for all $x \in X$. The fact that $H^0(f)$ is injective follows from $0 = s \in H^0(X, E)$ if and only if $s(x) = 0$ for all $x \in X$.

Let us now show that $\mathrm{im}(H^0(f)) = \ker(H^0(g))$. Since $H^0(g \circ f) = H^0(g) \circ H^0(f) = 0$, it follows, using Defintion 2.1.13 that $\mathrm{im}(H^0(f)) \subset \ker(H^0(g))$. Now assume $s \in \ker(H^0(g)) \subset H^0(X, E)$. We know from

$$0 \longrightarrow E'_x \xrightarrow{f_x} E_x \xrightarrow{g_x} E''_x$$

being exact that if $g_x(s(x)) = 0$, then $s(x) \in f_x(E'_x)$. Because $f : E' \to \mathrm{im}(f)$ is an isomorphism of vector bundles this shows that s factors through $f : E' \to E$ and so $s \in \mathrm{im}(H^0(f))$, i.e. we have $\ker(H^0(g)) \subset \mathrm{im}(H^0(f))$.

The following is a counter example to $H^0(g)$ being surjective.

Example 2.1.17. Consider a morphism of line bundles $f : \mathcal{O}(-2) \to \mathcal{O}$ on $\mathbb{P}^1$. Let $U = \{U_0, U_1\}$ be the standard open cover of $\mathbb{P}^1$ and let $g_{01} = \frac{z_0^2}{z_1^2}$ and $g'_{01} = 1$ be transition functions of $\mathcal{O}(-2)$ and $\mathcal{O}$ respectively. Consider f to be the morphism given locally by f_i, where $f_0 : U_0 \to \mathbb{C}$ and $f_1 : U_1 \to \mathbb{C}$ are given by

$$f_0(z_0 : z_1) = \frac{z_1}{z_0}$$

and

$$f_1(z_0 : z_1) = \frac{z_0}{z_1}$$

Clearly these are holomorphic functions on U_1 and U_0, respectively. These holomorphic functions satisfy

$$g'_{01}(x) f_1(x) = f_0(x) g_{01}(x), \quad \forall x \in U_0 \cap U_1$$

Note that $f|_{\mathcal{O}(-2)_{(z_0:z_1)}}$ is an isomorphism for all $(z_0 : z_1) \neq (0 : 1)$ or $(1 : 0)$. Using this morphism, we have an exact sequence in the category of coherent sheaves

$$0 \longrightarrow \mathcal{O}(-2) \xrightarrow{f} \mathcal{O} \longrightarrow \mathbb{C}_{P_1} \oplus \mathbb{C}_{P_2} \longrightarrow 0$$

where $P_1 = (0 : 1)$ and $P_2 = (1 : 0)$ and $\mathbb{C}_{P_i}$ is the skyscraper sheaf at P_i with stalk $\mathbb{C}$ for $i = 1, 2$. Now taking global sections we get an exact sequence

$$0 \longrightarrow H^0(\mathcal{O}(-2)) \xrightarrow{H^0(f)} H^0(\mathcal{O}) \xrightarrow{H^0(g)} H^0(\mathbb{C}_{P_1} \oplus \mathbb{C}_{P_2}) \longrightarrow \cdots$$

where $H^0(\mathcal{O}(-2)) = 0$ (this will be proved in Lemma 2.1.30 (a)). This gives us an exact sequence

$$0 \longrightarrow H^0(\mathcal{O}) \xrightarrow{H^0(g)} H^0(\mathbb{C}_{P_1} \oplus \mathbb{C}_{P_2}) \longrightarrow H^1(\mathcal{O}(-2)) \longrightarrow 0$$

where $H^0(\mathcal{O}) \cong \mathbb{C}$ and $H^0(\mathbb{C}_{P_1} \oplus \mathbb{C}_{P_2}) = H^0(\mathbb{C}_{P_1}) \oplus H^0(\mathbb{C}_{P_2}) = \mathbb{C} \oplus \mathbb{C}$, and so $H^0(g)$ could not be sujective.

Since we get the exact sequence (2.3) above, we know that the global section functor is left exact. The dimension of $H^i(X, E)$ will be denoted $h^i(X, E)$. In the case of a curve C the cohomology groups $H^i(C, E)$, vanish for all $i > 1$, where 1 is the dimension of C, i.e. only the cohomology groups $H^0(C, E)$ and $H^1(C, E)$ can be nonzero.

2.1.2 Ext groups

If E and E' are vector bundles over X, we denote by $\mathrm{Hom}_X(E, E')$ (or $\mathrm{Hom}(E, E')$ if it is clear which X we are referring to) the vector space of vector bundle morphisms. For a fixed E, $\mathrm{Hom}(E, \cdot)$ is a left exact covariant functor from the category of vector bundles to the category of vector spaces, i.e. given a short exact sequence of vector bundles

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

we get another exact sequence in which the last map is not surjective in general

$$0 \longrightarrow \mathrm{Hom}(E, F') \longrightarrow \mathrm{Hom}(E, F) \longrightarrow \mathrm{Hom}(E, F'') \tag{2.4}$$

We know from Theorem 1.4.67 that we can extend our short exact sequence (2.4) to a long exact sequence as follows:

$$\begin{array}{cccccccc} 0 & \longrightarrow & \mathrm{Hom}(E, F') & \longrightarrow & \mathrm{Hom}(E, F) & \longrightarrow & \mathrm{Hom}(E, F'') \\ & \longrightarrow & \mathrm{Ext}^1(E, F') & \longrightarrow & \mathrm{Ext}^1(E, F) & \longrightarrow & \cdots \end{array}$$

where $\mathrm{Ext}^i(E, \cdot)$ are the right derived functors of $\mathrm{Hom}(E, \cdot)$. So in particular we have $\mathrm{Ext}^0(E, \cdot) = \mathrm{Hom}(E, \cdot)$. We have the following proposition to see the relationship between the cohomology groups H^i and the Ext groups.

Proposition 2.1.18. *(a) For any locally free sheaf $\mathscr{L}$ and any sheaf $\mathscr{M}$ on a complex manifold X, we have:*
$\mathrm{Ext}^i(\mathscr{L}, \mathscr{M}) \cong H^i(\mathscr{L}^* \otimes \mathscr{M})$ *for all* $i \geq 0$.
(b) For any vector bundle E on a complex manifold X, we have:
$\mathrm{Ext}^i(\mathcal{O}_X, E) \cong H^i(E)$ *for all* $i \geq 0$.
Similarly we have:
$\mathrm{Ext}^i(E, \mathcal{O}_X) \cong H^i(E^*)$ *(where E^* is the dual bundle of E) for all* $i \geq 0$.

Proof. ([H] Proposition III.6.3) To begin with, we will prove the first statement of part (b):

First let us examine the case where $i = 0$. Let $f \in \mathrm{Hom}(\mathcal{O}, E)$, a homomorphism of vector bundles such that the following diagram commutes

$$\begin{tikzcd} X \times \mathbb{C} \arrow[rr, "f"] \arrow[dr, "\mathrm{pr}_1"'] & & E \arrow[dl, "p"] \\ & X & \end{tikzcd} .$$

Let $s : X \to E$ be a holomorphic section of E, i.e. $p \circ s = \mathrm{id}_X$. Define two linear maps

$$\alpha : \mathrm{Hom}(\mathcal{O}, E) \to H^0(E)$$

and

$$\beta : H^0(E) \to \mathrm{Hom}(\mathcal{O}, E)$$

as follows: Define $\alpha(f)(x) := f(x, 1)$ and $\beta(s)(x, \lambda) := \lambda \cdot s(x)$.

We see that $\beta(\alpha(f)) = f$ as follows:

$\beta(\alpha(f))(x, \lambda) = \lambda \cdot (\alpha(f)(x)) = \lambda \cdot f(x, 1) = f(x, \lambda)$, where the last equality uses the fact that f is linear on fibres.

Similarly we obtain $\alpha(\beta(s))(x) = \beta(s)(x, 1) = s(x)$, so $\alpha(\beta(s)) = s$. Hence α and β are inverses of one another and we get

$$\mathrm{Hom}(\mathcal{O}, E) \cong H^0(E).$$

Now we want to show that this isomorphism is functorial, i.e. that the following diagram commutes for all $\varphi : E \to F$, morphisms of vector bundles

$$\begin{array}{ccc} \mathrm{Hom}(\mathcal{O}, E) & \xrightarrow{\alpha_E} & H^0(E) \\ {\scriptstyle \mathrm{Hom}(\mathcal{O},\varphi)}\downarrow & & \downarrow{\scriptstyle H^0(\varphi)} \\ \mathrm{Hom}(\mathcal{O}, F) & \xrightarrow{\alpha_F} & H^0(F) \end{array}$$

Now if $f \in \mathrm{Hom}(\mathcal{O}, E)$, then $H^0(\varphi)(\alpha_E(f))(x) = \varphi(f(x, 1))$ (as in general $H^0(\alpha)(s)(x) = \alpha(s(x))$ for all $s \in H^0(E)$). On the other hand we get $\alpha_F(\mathrm{Hom}(\mathcal{O}, \varphi)(f))(x) = \alpha_F(\varphi \circ f)(x) = (\varphi \circ f)(x, 1) = \varphi(f(x, 1))$.

This allows us to conclude that since $\mathrm{Ext}^i(\mathcal{O}, -)$ is the i-th derived functor of $\mathrm{Hom}(\mathcal{O}, -)$ and $H^i(-)$ is the i-th derived functor of $H^0(-)$ that $\mathrm{Ext}^i(\mathcal{O}, E) \cong H^i(E)$ for all $i \geq 0$ and for any vector bundle E on X.

(a) Let us look at the case where $i = 0$. So we want $\mathrm{Hom}(\mathscr{L}, \mathscr{M}) \cong H^0(\mathscr{L}^* \otimes \mathscr{M})$.

We know $\mathrm{Hom}(\mathscr{L} \otimes \mathcal{O}, \mathscr{M}) \cong \mathrm{Hom}(\mathscr{L}, \mathscr{M})$. From [H] Exercise II.5.1 (b) and (c) we get $\mathrm{Hom}(\mathscr{L} \otimes \mathcal{O}, \mathscr{M}) \cong \mathrm{Hom}(\mathcal{O}, \mathscr{L}^* \otimes \mathscr{M})$. Then from the first statement of (b) above we get

$$\mathrm{Hom}(\mathscr{L}, \mathscr{M}) \cong H^0(\mathscr{L}^* \otimes \mathscr{M}).$$

Now, $\mathrm{Ext}^i(\mathscr{L}, -)$ is the i-th derived functor of $\mathrm{Hom}(\mathscr{L}, -)$ and $H^i(-)$ is the i-th derived functor of $H^0(-)$. Hence, $\mathrm{Ext}^i(\mathscr{L}, \mathscr{M}) \cong H^i(\mathscr{L}^* \otimes \mathscr{M})$ for all $i \geq 0$.

(b) The statement $\mathrm{Ext}^i(E, \mathcal{O}) \cong H^i(E^*)$ is a special case of (a), where $\mathscr{M} = \mathcal{O}$ and the vector bundle E corresponds to the locally free sheaf $\mathscr{L}$. □

The notion of a degree of a line bundle was introduced in Section 1.2. We can extend this to the degree of a vector bundle. To do so we must first define the determinant line bundle.

Definition 2.1.19. Given an open cover $\{U_i\}$ of X and a vector bundle E on X with transition functions g_{ij}, we have the following:

$$\begin{tikzcd} g_{ij} : U_i \cap U_j \arrow[r] \arrow[dr, "\det(g_{ij})"'] & \mathrm{GL}(n, \mathbb{C}) \arrow[d, "\det"] \\ & \mathrm{GL}(1, \mathbb{C}) \end{tikzcd}$$

with det being the determinant of the matrix and $\det(g_{ij})$ being the induced map on the transition functions g_{ij}. The *determinant line bundle* of E, denoted $\det E$ is given by transition functions h_{ij} where

$$h_{ij}(x) := \det g_{ij}(x) \in \mathrm{GL}(1, \mathbb{C}), \quad \text{for all } x \in U_i \cap U_j.$$

Let us see that this really is a line bundle. So we have $h_{ij} : U_i \cap U_j \to \mathrm{GL}(1, \mathbb{C})$. We must check the following conditions:

$$h_{ij}(x) \cdot h_{ji}(x) = 1 \qquad \text{for all } x \in U_i \cap U_j$$

$$h_{ij}(x) \cdot h_{jk}(x) \cdot h_{ki}(x) = 1 \qquad \text{for all } x \in U_i \cap U_j \cap U_k.$$

We know

$$g_{ij}(x) \circ g_{ji}(x) = \mathrm{id} \qquad \text{for all } x \in U_i \cap U_j \tag{2.5}$$

$$g_{ij}(x) \circ g_{jk}(x) \circ g_{ki}(x) = \mathrm{id} \qquad \text{for all } x \in U_i \cap U_j \cap U_k. \tag{2.6}$$

Applying det to (2.5) we get

$$\det(g_{ij}(x) \circ g_{ji}(x)) = \det(\mathrm{id}).$$

which gives

$$\det(g_{ij}(x)) \cdot \det(g_{ji}(x)) = 1.$$

Similarly applying det to (2.6) we get

$$\det(g_{ij}(x) \circ g_{jk}(x) \circ g_{ki}(x)) = \det(\mathrm{id})$$

which gives

$$\det(g_{ij}(x)) \cdot \det(g_{jk}(x)) \cdot \det(g_{ki}(x)) = 1.$$

and so the functions h_{ij} are really transition functions of a line bundle.

This now allows us to define the degree of a vector bundle as follows.

Definition 2.1.20. The *degree* $\deg E \in \mathbb{Z}$ of a vector bundle, E, is the degree of its determinant line bundle $\det E$ (See Definition 1.2.18 for the definition of the degree of a line bundle).

If E lies in an exact sequence of holomorphic vector bundles on X, with open cover $\{U_i\}$ as follows:

$$0 \to E' \to E \to E'' \to 0$$

with transition functions of E', E and E'' being $g'_{ij}, g_{ij}, g''_{ij}$, respectively, such that all three bundles are trivial on each U_i, then from Definition 2.1.4, we have the following

$$g_{ij}(x) = \begin{pmatrix} g'_{ij}(x) & * \\ 0 & g''_{ij}(x) \end{pmatrix}.$$

and taking determinants we get

$$\det(g_{ij}) = \det(g'_{ij}) \cdot \det(g''_{ij})$$

and so from Defintion 1.2.10 we get an isomorphism

$$\det E \cong \det E' \otimes \det E''.$$

Since $\deg : \mathrm{Pic}(X) \to \mathbb{Z}$ is a homomorphism, we get $\deg E = \deg E' + \deg E''$. In other words, degree is additive on exact sequences.

As a special case, if a vector bundle, $E = L_1 \oplus L_2$, is the direct sum of two line bundles L_1 and L_2, then we have

$$0 \longrightarrow L_1 \xrightarrow[i_1]{} E \xrightarrow[\mathrm{pr}_2]{} L_2 \longrightarrow 0$$

with i_1 being the natural inclusion and pr_2 being the projection to the second factor, and so we get $\deg E = \deg L_1 + \deg L_2$.

Remark 2.1.21. Let E be a vector bundle or rank r and L be a line bundle over a complex manifold X, with an open $\{U_i\}$ of X such that E and L are trivial on each U_i. Let E and L have transition functions g_{ij} and h_{ij}, respectively. If we tensor E with L, we know from Definition 2.1.9 that the transition functions of $E \otimes L$ is

$$k_{ij}(x) := h_{ij}(x) \cdot g_{ij}(x) \text{ for all } x \in U_i \cap U_j.$$

Taking det we have $\det(h_{ij}(x) \cdot g_{ij}(x)) = h^r_{ij}(x) \cdot \det g_{ij}(x)$. From this we get $\deg(E \otimes L) = r \deg L + \deg E$. In particular if E is a vector bundle of rank 2 we have $\deg(E \otimes L) = \deg E + 2 \deg L$.

For this reason, when considering vector bundles of rank 2, it is enough to consider vector bundles of degree -1 or 0, (or indeed vector bundles of any even and odd degree), then by tensoring with a line bundle of appropriate degree we get all other degrees. We call this the “tensor product trick”.

More generally, if E and E' are vector bundles over a complex manifold X of rank r and r' respectively, then $\deg(E \otimes E') = r' \deg E + r \deg E'$.

Definition 2.1.22. An exact sequence of vector bundles

$$0 \to E' \to E \to E'' \to 0$$

splits if and only if there exists a morphism $f : E'' \to E$ for which the composition $E'' \xrightarrow{f} E \longrightarrow E''$ is an isomorphism, or equivalently if there exists a morphism $g : E \to E'$ for which the composition $E' \hookrightarrow E \xrightarrow{g} E'$ is an isomorphism.

In this case, either of the maps f or g is called a *splitting* of the sequence. If the sequence above splits then $E \cong E' \oplus E''$.

Now consider, on any curve, a short exact sequence of vector bundles

$$\mathbb{E}: \quad 0 \longrightarrow M \xrightarrow{\alpha} E \xrightarrow{\beta} L \longrightarrow 0.$$

By applying $F = \mathrm{Hom}(L, -)$ to this sequence we get the following morphism:

$$\mathrm{Hom}(L, L) \xrightarrow{\delta} \mathrm{Ext}^1(L, M)$$

Definition 2.1.23. The image under the coboundary map δ of the identity of L, $\mathrm{id}_L \in \mathrm{Hom}(L, L)$, which we will denote by

$$\delta(\mathrm{id}_L) \in \mathrm{Ext}^1(L, M) \cong H^1(L^* \otimes M),$$

is called the *extension class* of $\mathbb{E}$.

By exactness of

$$\mathrm{Hom}(L, E) \xrightarrow{F(\beta)} \mathrm{Hom}(L, L) \xrightarrow{\delta} \mathrm{Ext}^1(L, M),$$

if $\delta(\mathrm{id}_L) = 0$, then there exists a morphism $f : L \to E$ for which the composition $\mathrm{id}_L = \beta \circ f : L \to L$, i.e. the sequence $\mathbb{E}$ splits. Moreover, if $\mathbb{E}$ splits, there exists $f : L \to E$ such that $\mathrm{id}_L = \beta \circ f$. Because the composition $F(\beta) \circ \delta$ is zero, we have $\delta(\mathrm{id}_L) = 0$. Hence we have the following proposition:

Proposition 2.1.24. *If* $\mathrm{Ext}^1(L, M) \cong H^1(L^* \otimes M) = 0$*, then every exact sequence* $\mathbb{E}$ *splits.*

Remark 2.1.25. ([W] Section 3.4) For each $\alpha \in \mathrm{Ext}^1(E'', E')$ there exists an extenstion

$$0 \to E' \to E_\alpha \to E'' \to 0 \tag{2.7}$$

with a vector bundle, E_α, in such a way that α is the extension class of (2.7). Moreoever, $E_\alpha \cong E_\beta$ if and only if there exists $\lambda \in \mathbb{C}^*$ such that $\alpha = \lambda\beta$.

2.1.3 Riemann-Roch formula for curves

We have a very useful tool, called the Riemann-Roch formula, which tells us alot about the cohomology groups, $H^0(E)$ and $H^1(E)$, once we know the rank and degree of the vector bundle E. In order to state the Riemann-Roch formula, we need to know what the genus of a curve is. The genus of a complex manifold is an important topological invariant. Let us give a precise definition in the case of a curve.

Definition 2.1.26. The *genus*, g, of a curve, C, is

$$g := \mathrm{h}^1(\mathcal{O}_C).$$

The one-dimensional projective space, $\mathbb{P}^1$, has $g = 0$. An elliptic curve $\mathbb{C}/\Lambda$ (as described in Example 1.1.7) has $g = 1$.

We can now give the Riemann-Roch formula for curves as follows: If E is a vector bundle, of rank r on a curve of genus g, then:

$$h^0(E) - h^1(E) = \deg E + r(1 - g).$$

In addition to the Riemann-Roch formula, one of the other major tools we have in dealing with cohomology is Serre duality. First we need to define the canonical line bundle on a curve, C.

Definition 2.1.27. Let C be a curve, i.e. a one-dimensional complex manifold. Let $\{U_i\}$ be an open cover of C and let $\varphi_i : U_i \to \mathbb{C}$ be coordinate maps. Using the notation of Remark 1.1.3, we know $V_{ij} \subset \mathbb{C}$ for all i, j, since C is a curve. Recall that $\varphi_i \circ \varphi_j^{-1} =: \psi_{ij} : V_{ji} \to V_{ij}$.

The *canonical line bundle*, K_C of C is given by transition functions, g_{ij} where

$$g_{ij}(x) := \psi'_{ij}(\varphi_j(x)) \in \mathrm{GL}(1, \mathbb{C}), \text{for all } x \in U_i \cap U_j,$$

i.e. g_{ij} is the derivative of the change of coordinate maps of C. Let us check that this defines a line bundle as outlined in Definition 1.2.6. So we have $g_{ij} : U_i \cap U_j \to \mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$. We must check that

$$g_{ij}(x) \cdot g_{ji}(x) = 1 \text{ for all } x \in U_i \cap U_j. \tag{2.8}$$

$$g_{ij}(x) \cdot g_{ji}(x) \cdot g_{ki}(x) = 1 \text{ for all } x \in U_i \cap U_j \cap U_k. \tag{2.9}$$

We know the change of coordinate maps of C satisfy the above identities, i.e.

$$\psi_{ij}(\psi_{ji}(z)) = z \text{ for all } z \in V_{ij}$$

and

$$\psi_{ij}(\psi_{jk}(\psi_{ki}(z))) = z \text{ for all } z \in V_{ij} \cap V_{ik}.$$

(see again Remark 1.1.3) so taking derivatives of these we get

$$\psi'_{ij}(\psi_{ji}(z)) \cdot \psi'_{ji}(z) = \mathrm{id}$$

where $z = \varphi_i(x)$, with $x \in U_i \cap U_j$. This gives

$$\psi'_{ij}(\psi_{ji}(\varphi_i(x)) \cdot \psi'_{ji}(\varphi_i(x)) = \mathrm{id}$$

since by definition, $\psi_{ij}(\varphi_i(x)) = \varphi_j(x)$, we have (2.8) (by definition of g_{ij}). Similarly we can show (2.9) and so the g_{ij} really define a line bundle, K_C.

Example 2.1.28. Let $C = \mathbb{P}^1$ and let $\{U_0, U_1\}$ be the standard open cover of $\mathbb{P}^1$ and $z = (z_0 : z_1)$, a point in $\mathbb{P}^1$. Using the complex structure as outlined in Example 1.1.6, we know that the change of coordinate maps for $\mathbb{P}^1$ are $\psi_{01}(z) = \frac{1}{z}$ and so $K_{\mathbb{P}^1}$ is given by transition functions $g_{01}(z) = (\frac{1}{z})' = -z^{-2}$. From Example 1.2.9 and Remark 2.1.8 we know this is isomorphic to the line bundle $\mathcal{O}_{\mathbb{P}^1}(-2)$.

Outlined below is some information about K_C is in the case of a curve:

$$C = \mathbb{P}^1 : \quad K_C = \mathcal{O}_{\mathbb{P}^1}(-2), \quad \deg(K_C) = -2$$

$$C = \text{elliptic curve} : \quad K_C = \mathcal{O}_C, \quad \deg(K_C) = 0$$

$$C = \text{curve of genus } g \geq 2 : \quad \deg(K_C) = 2g - 2.$$

The following proposition outlines Serre duality, though it will not be proved as the proof is too involved for this paper (further details can be found in [H] Section III.7).

Proposition 2.1.29. (**Serre duality**) *Let C be a smooth projective curve and E be a vector bundle on C. Let K_C be the canonical line bundle on C. Then there are isomorphisms that are functorial in E*

$$H^0(C, E) \cong H^1(C, K_C \otimes E^*)^*.$$

and

$$H^1(C, E) \cong H^0(C, K_C \otimes E^*)^*.$$

In particular it follows that $H^0(C, E)$ and $H^1(C, K_C \otimes E^*)$ have the same dimension.

Lemma 2.1.30. *Let L be a line bundle on a curve, C, of genus g. Then we have the following:*

(a) $H^0(L) = 0$ if $\deg L < 0$.
(b) $H^1(L) = 0$ if $\deg L > 2g - 2$.
(c) $L \cong \mathcal{O}$ if $\deg L = 0$ and $H^0(L) \neq 0$.

Proof. (a) Assume $H^0(L) \neq 0$, so we have $s \in H^0(L), s \neq 0$ with $\operatorname{div}(s) = \sum_{P \in C} \operatorname{ord}_P(s) \cdot P$. Now s is a holomorphic section so $\operatorname{ord}_P(s) \geq 0$ for all $P \in C$. Hence, $\deg(\operatorname{div}(s)) = \sum \operatorname{ord}_P(s) \geq 0$. Now $\mathcal{O}(\operatorname{div}(s)) \cong L$. But $\deg L < 0$ and $\deg(\operatorname{div}(s)) \geq 0$ so we have a contradiction. Hence, $H^0(L) = 0$.

(b) From Serre duality we know $H^1(L) \cong H^0(L^* \otimes K_C)^*$. Now $\deg(L^* \otimes K_C) = 2g - 2 - \deg L$. Since $\deg L > 2g - 2$, we get $\deg(L^* \otimes K_C) < 0$ and from part (a) above we get $H^0(L^* \otimes K_C) \cong H^1(L) = 0$.

(c) Let $s \in H^0(L)$ be a nonzero section in L. Let $D = \operatorname{div}(s)$, the $L \cong \mathcal{O}(D)$. We know that

$$\operatorname{div}(s) = \sum_P \operatorname{ord}_P(s) \cdot P.$$

Since we know $\deg L = 0$, then we have $\sum_P \operatorname{ord}_P(s) = 0$. This implies that $\operatorname{ord}_P(s) = 0$ for all $P \in C$. This in turn implies that $D = 0$. So we have $L = \mathcal{O}(D) \cong \mathcal{O}$. □

Lemma 2.1.31. *If E is an indecomposable vector bundle of rank 2 on a smooth projective curve, C, then every line subbundle $L \subset E$ satisfies*

$$2 \deg L \leq \deg E + 2g - 2$$

Proof. ([Mu] Lemma 10.40) Let M be the quotient line bundle E/L. This gives us the following short exact sequence:

$$0 \to L \to E \to M \to 0$$

which corresponds to an element in $\operatorname{Ext}^1(M, L) \cong H^1(M^* \otimes L)$. Now since E is indecomposable, this sequence cannot split and hence $H^1(M^* \otimes L) \neq 0$ by Proposition 2.1.24. By Serre duality, this implies that $0 \neq H^0((M^* \otimes L)^* \otimes K_C)^* = H^0(M \otimes L^* \otimes K_C)^*$. This in turn implies that $\deg(M \otimes L^* \otimes K_C) = \deg M - \deg L + 2g - 2 \geq 0$ from Lemma 2.1.30 (a). Now from the short exact sequence above we know that $\deg E = \deg M + \deg L$, i.e. $\deg M = \deg E - \deg L$. Hence we get

$$\deg E - \deg L - \deg L + 2g - 2 \geq 0$$

From this, we get the inequality in the lemma. □

Lemma 2.1.32. *If E is a vector bundle on a curve C of genus g, then the degree of its subbundles $F \subset E$ is bounded above.*

Proof. ([Mu], Corollary 10.9) Since the global sections functor is left exact, for each subbundle $F \subset E$ we get

$$H^0(F) \subset H^0(E)$$

This implies that $h^0(F) \leq h^0(E)$. Now by Riemann-Roch we know

$$h^0(F) - h^1(F) = \deg(F) + \mathrm{rk}(F) \cdot (1 - g).$$

$$\deg(F) = h^0(E) - \mathrm{rk}(F) \cdot (1 - g) - h^1(F)$$

Now if $g = 1$, we see that $\deg(F) = h^0(E) - h^1(F)$ and since $h^1(F) \geq 0$, we get $\deg(F) \leq h^0(E)$.

If $g = 0$, then $\deg(F) = h^0(E) - \mathrm{rk}(F) - h^1(F)$ and since $\mathrm{rk}(F) \geq 0$ and $h^1(F) \geq 0$, we see that $\deg(F) \leq h^0(E)$.

If $g \geq 2, \deg(F) = h^0(E) + \mathrm{rk}(F) \cdot (g - 1) - h^1(F) \leq h^0(E) + \mathrm{rk}(E) \cdot (g - 1) - h^1(F)$ (since $\mathrm{rk}(F) \leq \mathrm{rk}(E)$). Again, since $h^1(F) \geq 0$, we get $\deg(F) \leq h^0(E) + \mathrm{rk}(E) \cdot (g - 1)$. Hence we see that in any case the degree of $F \subset E$ is bounded above. $\square$

2.2 Vector bundles on $\mathbb{P}^1$

Before we move on to vector bundles on an elliptic curve (i.e. a curve of genus one), it makes sense to look at vector bundles on a curve of genus zero ($\mathbb{P}^1$). Let us now restate Lemma 2.1.30 in the case of $\mathbb{P}^1$, where genus $g = 0$, to see how the cohomology of line bundles on $\mathbb{P}^1$ is particularly simple.

Lemma 2.2.1. *Let L be a line bundle on a $\mathbb{P}^1$. Then we have the following:*
(a) $H^0(L) = 0$ if $\deg L \leq -1$.
(b) $H^1(L) = 0$ if $\deg L \geq -1$.
(c) $L \cong \mathcal{O}$ if $\deg L = 0$ and $H^0(L) \neq 0$.

By Riemann-Roch we also have,

$$h^0(L) - h^1(L) = \deg L + 1.$$

Lemma 2.2.2. $\deg : \mathrm{Pic}(\mathbb{P}^1) \to \mathbb{Z}$ *is an isomorphism.*

Proof. ([G] Lemma 6.3.11) The morphism deg is clearly surjective: for any $n \in \mathbb{Z}$, there is a line bundle in $\mathrm{Pic}(\mathbb{P}^1)$ of degree n, namely $\mathcal{O}(n)$ (See Example 1.2.9). So we must now show that deg is injective, i.e. that $\ker(\deg) = \mathcal{O}$. We know that $\ker(\deg) = \{L \in \mathrm{Pic}(\mathbb{P}^1) | \deg L = 0\}$. Now if $\deg L = 0, L \in \mathrm{Pic}(\mathbb{P}^1)$, then by Riemann-Roch we have $h^0(L) - h^1(L) = 1$. Hence $H^0(L) \neq 0$ and so by Lemma 2.2.1 (c) we know $L \cong \mathcal{O}$, i.e. deg is injective and thus $\deg : \mathrm{Pic}(\mathbb{P}^1) \to \mathbb{Z}$ is an isomorphism. □

We have a classification for all vector bundles on $\mathbb{P}^1$ as follows:

Lemma 2.2.3. *Every rank* 2 *vector bundle on* $\mathbb{P}^1$ *is isomorphic to a direct sum of two line bundles*

Proof. ([Mu], Lemma 10.30) Let E be a rank 2 vector bundle on $\mathbb{P}^1$. Tensoring with a line bundle if necessary, it is enough to assume that $\deg E = 0$ or -1. First, the Riemann-Roch formula tells us that $h^0(E) - h^1(E) = \deg(E) + 2$. Since $\deg E = 0$ or -1 we have $H^0(E) \neq 0$. From Proposition 2.1.18, this gives us a nonzero morphism of sheaves $f : \mathcal{O} \to E$. So we get the following short exact sequence:

$$0 \longrightarrow \ker f \longrightarrow \mathcal{O} \xrightarrow{f} \operatorname{im} f \longrightarrow 0 .$$

Because $\ker f \subset \mathcal{O}$ and $\operatorname{im} f \subset E$, both are locally free (see Remark 1.3.22). Now $\mathrm{rk}(\mathcal{O}) = 1 = \mathrm{rk}(\ker f) + \mathrm{rk}(\operatorname{im} f)$. If $\mathrm{rk}(\ker f) = 1$, then we would have $\mathrm{rk}(\operatorname{im} f) = 0$, which would imply that $f = 0$, a contradiction. So $\mathrm{rk}(\ker f) = 0$. This implies that $\ker f = 0$, i.e. f is a monomorphism in the category of coherent sheaves. Therefore, we can apply saturation (Remark 2.1.16) to get a commutative diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r, "f"] \arrow[d, "\alpha"] & E \arrow[r, "g"] \arrow[d, equal] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r, "f'"'] & E \arrow[r, "g'"'] & L \arrow[r] & 0
\end{tikzcd}$$

with $M \subset E$, a line subbundle and $L := E/M$, a line bundle. Now since $f \neq 0$, then we know $0 \neq \alpha \in \mathrm{Hom}(\mathcal{O}, M) = H^0(M)$. This implies (from Lemma 2.2.1 (a)) that $\deg(M) \geq 0$. Now $\deg E = \deg L + \deg M$, hence $\deg(L^* \otimes M) = \deg M - \deg L = -\deg E + 2\deg M \geq -\deg E \geq 0$. From Lemma 2.2.1 (b), we get $H^1(L^* \otimes M) = 0$. By Proposition 2.1.24, therefore, the sequence $0 \to M \to E \to L \to 0$ splits. $\square$

Grothendieck's Theorem 2.2.4. *Every vector bundle on $\mathbb{P}^1$ is isomorphic to a direct sum of line bundles*

Proof. ([Mu] Theorem 10.31) Let E be a vector bundle of rank r on $\mathbb{P}^1$. Proof is by induction on the rank $r \geq 2$ of E, starting with the previous lemma. Serre's Theorem ([H] II.5.17) tells us that there exists a line subbundle in E. Now let $M \subset E$ be a line subbundle whose degree, $m = \deg M$, is maximal among line subbundles of E (Lemma 2.1.32). Now $F := E/M$ is a vector bundle of rank $r - 1$.

Claim: Every line subbundle $L \subset F$ has $\deg L \leq m$.

Now we have a short exact sequence as follows:

$$0 \to M \to E \to F \to 0$$

By considering the preimage $\tilde{L} \subset E$ of L under the quotient morphism $E \to F$ we get a diagram as follows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] & E \arrow[r] & F \arrow[r] & 0 \\
0 \arrow[r] & M \arrow[r] \arrow[u, equal] & \tilde{L} \arrow[r] \arrow[u, hook] & L \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

Clearly $\tilde{L}$ is a rank 2 vector bundle, and we see $\deg \tilde{L} = m + \deg L$. By Lemma 2.2.3, we know $\tilde{L} \cong L_1 \oplus L_2$ for some line bundles L_1 and L_2 on $\mathbb{P}^1$. Now $\deg(\tilde{L}) = \deg(L_1) + \deg(L_2)$ so one of L_1 or L_2 must have deg at least $\deg(\tilde{L})/2$. Let N denote that line subbundle, of degree at least $\deg \tilde{L}/2$. Because N is a subbundle of E, as well as our choice of M we get $m \geq \deg N \geq (\deg \tilde{L})/2 = \frac{m+\deg L}{2}$ and the claim follows easily from this.

By the inductive hypothesis, we know the quotient bundle F is isomorphic to a direct sum $F = L_1 \oplus \cdots \oplus L_{r-1}$ of line bundles and the claim gives us $\deg L_i \leq m$. Since $\deg(L_i^* \otimes M) = \deg M - \deg L_i = m - \deg L_i \geq 0$, we get $H^1(L_i^* \otimes M) = 0$ for each i. It follows from Proposition 2.1.24 that the exact sequence

$$0 \to M \to E \to \bigoplus_{i=1}^{r-1} L_1 \to 0$$

splits. $\square$

Definition 2.2.5. A vector bundle, E, is called *decomposable* if it is isomorphic to the direct sum $E_1 \bigoplus E_2$ of two nonzero vector bundles; otherwise, E is called *indecomposable*.

By definition of decomposability, every vector bundle is the direct sum of indecomposable ones. Therefore, it suffices to know the indecomposable vector bundles on a curve in order to know them all. We have seen that all vector bundles on rational curves are the direct sum of line bundles. As well as the notion of an indecomposable vector bundle, we also have the notion of a simple vector bundle.

Definition 2.2.6. A vector bundle E is *simple* if its only endomorphisms are scalars, $\operatorname{End} E = \mathbb{C}$.

A simple vector bundle is necessarily indecomposable. To see this let us start with a decomposable vector bundle $E \oplus F$. Consider $f : E \oplus F \to E \oplus F$, where $f = \mathrm{id}_E \oplus 0_F$ where id_E is the identity map on E and 0_F is the zero map on F. Clearly this morphism is not a multiple of the identity and so $\operatorname{End}(E \oplus F) \neq \mathbb{C}$, i.e. F is not simple.

Note that the converse is not true, i.e. an indecomposable vector bundle is not necessarily simple (This can be seen by a counter example, Example 2.3.3).

2.3 Rank two vector bundles on an elliptic curve

We are now ready to look at the case of a nonsingular curve of arithmetic genus one (i.e. an elliptic curve). Atiyah's paper of 1957 ([At]) provided us with an answer to this case. We have already seen in Lemma 2.2.2 that there is exactly one line bundle on $\mathbb{P}^1$ for every degree. In particular $\mathrm{Pic}^0(\mathbb{P}^1) = \{\mathcal{O}\}$, where $\mathrm{Pic}^0(\mathbb{P}^1)$ denotes the set of line bundles of degree 0 on $\mathbb{P}^1$. However, it turns out on an elliptic curve, C, that $\mathrm{Pic}^0(C)$ is in bijection to C (Theorem 2.3.7) and so on elliptic curves there are more vector bundles in the sense that nontrivial extensions appear. In this section we will be concentrating on rank 2 vector bundles on an elliptic curve. We will now give a classification of all indecomposable rank 2 vector bundles on an elliptic curve C.

First let me return to the Riemann-Roch formula for a vector bundle E, this time looking at a curve of genus 1, i.e.

$$h^0(E) - h^1(E) = \deg E$$

Note that every line bundle, L, on C satisfies:

$$h^0(L) - h^1(L) = \deg L$$

Moreover, we have the following lemma (as a particular case of Lemma 2.1.30):

Lemma 2.3.1. *Let L be a line bundle on an elliptic curve. Then we have the following:*

(a) $H^0(L) = 0$ if $\deg L < 0$.

(b) $H^1(L) = 0$ if $\deg L > 0$.

(c) If $\deg L = 0$ and $L \ncong \mathcal{O}$, then $H^0(L) = H^1(L) = 0$.

Let $\mathcal{E}(r, d)$ denote the set of isomorphism classes of indecomposable vector bundles of rank r and degree d over C, an elliptic curve.

Theorem 2.3.2. *(a) There exists a vector bundle $E_r \in \mathcal{E}(r, 0)$, for each integer $r \geq 1$, unique up to isomorphism, with $H^0(E_r) \neq 0$. Moreover, we have a nonsplit exact sequence:*

$$0 \to \mathcal{O}_C \to E_r \to E_{r-1} \to 0$$

(b) Let $E \in \mathcal{E}(r, 0)$, then $E \cong E_r \otimes L$, where L is a line bundle of degree zero, unique up to isomorphism.

Proof. See [At] Theorem 5. □

Example 2.3.3. The bundles E_r of Theorem 2.3.2 are sometimes called the Atiyah bundles. For $r \geq 2$, they are examples of indecomposable vector bundles which are not simple. Let us prove now that E_2 is not simple.

We know E_2 sits in an exact sequence as follows:

$$0 \longrightarrow \mathcal{O}_C \xrightarrow{f} E_2 \longrightarrow \mathcal{O}_C \longrightarrow 0$$

Applying the contravariant functor $\mathrm{Hom}(-, E_2)$ to this sequence, we get

$$0 \longrightarrow \mathrm{Hom}(\mathcal{O}, E_2) \longrightarrow \mathrm{Hom}(E_2, E_2) \xrightarrow{\beta} \mathrm{Hom}(\mathcal{O}, E_2)$$

Now $\mathrm{Hom}(\mathcal{O}, E_2) \cong H^0(E_2)$ from Proposition 2.1.18. From our assumption on E_2, we know $H^0(E_2) \neq 0$, i.e. $h^0(E_2) \geq 1$. Let id_{E_2} denote $\mathrm{id} \in \mathrm{Hom}(E_2, E_2)$. We know $\beta(\mathrm{id}_{E_2}) = f \neq 0$. This implies $\beta \neq 0$. So we get the following short exact sequence:

$$0 \to \mathrm{Hom}(\mathcal{O}, E_2) \to \mathrm{Hom}(E_2, E_2) \to \mathrm{im}(\beta) \to 0$$

Since $\beta \neq 0$, we know that $\dim(\mathrm{im}(\beta)) \geq 1$. Now since dim is additive on exact sequences, we have $\dim(\mathrm{Hom}(E_2, E_2)) = \dim(\mathrm{Hom}(\mathcal{O}, E_2)) + \dim(\mathrm{im}(\beta)) \geq 2$. Hence by the definition of a simple vector bundle (Definition 2.2.6), we know that E_2 is not simple.

Let us now classify all indecomposable rank 2 vector bundles on an elliptic curve. We first consider the case of odd degree.

Proposition 2.3.4. *On a smooth curve, C, of genus 1, given a line bundle L of odd degree, there exists, up to isomorphism, a unique indecomposable rank 2 vector bundle E with $\det E \cong L$.*

Proof. ([M] Proposition 10.47) From Remark 2.1.21, it is enough to consider the case $\deg L = 1$. Since $\deg L = 1$, $\deg L^* = -1$ (as L^* is the inverse of the line bundle L in the Picard group and $\deg : \mathrm{Pic}(C) \to \mathbb{Z}$ is a homomorphism of groups) and from Lemma 2.3.1 then, we know that $H^0(L^*) = 0$. Riemann-Roch then tells us that $H^1(L^*)$ is 1-dimensional. Now $H^1(L^*) \cong \mathrm{Ext}^1(L, \mathcal{O})$ by Proposition 2.1.18 (c) and so from Remark 2.1.25 we know there is, up to isomorphism, just one vector bundle E which sits in a nonsplit short exact sequence as follows

$$0 \to \mathcal{O} \to E \to L \to 0. \tag{2.10}$$

Claim: $h^0(E) = 1$.

Since $H^0(L) \neq 0$, then we know $\mathrm{Hom}(\mathcal{O}, L) \neq 0$ and it follows that L contains $\mathcal{O}$ as a subsheaf. Consider the following diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r] & E \arrow[r, "f"] & L \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[u, equal] & E' \arrow[r] \arrow[u, hook] & \mathcal{O} \arrow[r] \arrow[u, hook, "\beta"] & 0
\end{tikzcd} \tag{2.11}$$

where $E' \subset E$ by the inverse image of this subsheaf, $\mathcal{O}$. Now consider the dual of the above diagram

$$\begin{tikzcd}
0 \arrow[r] & L^* \arrow[r] \arrow[d, hook, "\varphi"] & E^* \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O} \arrow[r] & E'^* \arrow[r] & \mathcal{O} \arrow[r] & 0
\end{tikzcd} \tag{2.12}$$

Let us show that $\varphi : L^* \to \mathcal{O}$ is a monomorphism. Because $\ker \varphi \subset L^*$ and $\mathrm{im}\, \varphi \subset \mathcal{O}$, both are locally free (see Remark 1.3.22). So we get an exact sequence of locally free sheaves

$$0 \to \ker \varphi \to L^* \to \mathrm{im}\, \varphi \to 0.$$

Now $\mathrm{rk}(L^*) = 1 = \mathrm{rk}(\ker\varphi) + \mathrm{rk}(\operatorname{im}\varphi)$. If $\mathrm{rk}(\ker\varphi) = 1$, then we would have $\mathrm{rk}(\operatorname{im}\varphi) = 0$, which would imply that $\varphi = 0$, a contradiction. So $\mathrm{rk}(\ker\varphi) = 0$. This implies that $\ker\varphi = 0$, i.e. φ is a monomorphism in the category of coherent sheaves.

Now note that the first row of (2.11) splits if and only if the first row of (2.12) splits, and the second row of (2.11) splits if and only if the second row of (2.12) splits. Applying $\mathrm{Hom}(\mathcal{O}, -) = H^0(-)$ to (2.12) we get the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & H^0(L^*) \arrow[r] \arrow[d] & H^0(E^*) \arrow[r] \arrow[d] & H^0(\mathcal{O}) \arrow[r, "\delta_3"] \arrow[d, equal] & H^1(L^*) \arrow[r] \arrow[d, "H^1(\varphi)"] & \cdots . \\ 0 \arrow[r] & H^0(\mathcal{O}) \arrow[r] & H^0(E'^*) \arrow[r] & H^0(\mathcal{O}) \arrow[r, "\delta_4"] & H^1(\mathcal{O}) \arrow[r] & \cdots \end{tikzcd} \tag{2.13}$$

Now since $\varphi : L^* \hookrightarrow \mathcal{O}$ is a monomorphism we have a short exact sequence as follows:

$$0 \longrightarrow L^* \xrightarrow{\varphi} \mathcal{O} \longrightarrow \mathscr{C} \longrightarrow 0$$

Clearly $\mathrm{rk}(\mathscr{C}) = 0$, i.e. $\mathscr{C}$ is a torsion sheaf. Hence $H^1(\mathscr{C}) = 0$. Now apply $H^0(-)$ to the short exact sequence above to get

$$0 \to H^0(\mathcal{O}) \to H^0(\mathscr{C}) \to H^1(L^*) \to H^1(\mathcal{O}) \to 0$$

Hence $H^1(\varphi) : H^1(L^*) \to H^1(\mathcal{O})$ is a surjection between two vector spaces of the same dimension, hence an isomorphism (and in particular an injection).

Now consider $\mathrm{id} \in \mathrm{Hom}(\mathcal{O}, \mathcal{O}) = H^0(\mathcal{O})$. We know by assumption that the top row of (2.11) does not split. This implies that the top row of (2.12) does not split. This in turn implies that $\delta_3(\mathrm{id}) \neq 0$ (by Proposition 2.1.24). Now since $H^1(\varphi)$ is injective, we know that $\delta_4(\mathrm{id}) \neq 0$ which gives us that the second row of (2.12) does not split. Hence, the second row of (2.11), i.e.

$$0 \to \mathcal{O} \to E' \to \mathcal{O} \to 0$$

does not split. Applying $\mathrm{Hom}(\mathcal{O}, -) = H^0(-)$ to (2.11) we get the following commutative diagram with exact rows

$$
\begin{tikzcd}
0 \arrow[r] & H^0(\mathcal{O}) \arrow[r] & H^0(E) \arrow[r] & H^0(L) \arrow[r, "\delta_1"] & H^1(\mathcal{O}) \arrow[r] & \cdots . \\
0 \arrow[r] & H^0(\mathcal{O}) \arrow[r] \arrow[u, equal] & H^0(E') \arrow[r] \arrow[u] & H^0(\mathcal{O}) \arrow[r, "\delta_2"] \arrow[u, "f"] & H^1(\mathcal{O}) \arrow[r] \arrow[u, equal, "g"] & \cdots
\end{tikzcd}
\tag{2.14}
$$

with $\delta_2(\mathrm{id}) \neq 0$. Because $h^0(\mathcal{O}) = h^1(\mathcal{O}) = 1$, δ_2 is an isomorphism. Now $g \circ \delta_2 \neq 0$. This implies $\delta_1 \circ f \neq 0$. In particular $\delta_1 \neq 0$. Since $H^0(L)$ and $H^1(\mathcal{O})$ are one dimensional, we have δ_1 is an isomorphism. Hence $H^0(\mathcal{O}) \cong H^0(E)$. This gives us $h^0(E) = 1$.

We must now show the indecomposability of E. We know that $h^0(E) = 1$. Let $\phi \in \mathrm{End}(E)$, then we will denote by $H^0(\phi)$ the induced linear endomorphism of $H^0(E)$. Suppose that $H^0(\phi) = 0$. Then ϕ maps the line subbundle $\mathcal{O}$ to zero because $\mathcal{O}$ is generated by its global section 1. Therefore it factors through the quotient L:

$$\phi : E \to L \to E.$$

Now since (2.10) is nonsplit we have the following diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r, "\alpha"] & E \arrow[r, "\beta"] \arrow[d, "\phi"'] & L \arrow[r] \arrow[dl, "\gamma"] & 0 \\
 & & E \arrow[r, "\beta"'] & L &
\end{tikzcd}
$$

We know $\mathrm{Hom}(L, L) \cong \mathrm{Hom}(L^* \otimes L) \cong H^0(\mathcal{O})$ is one dimensional, i.e. $\mathrm{Hom}(L, L) = \mathrm{id}_L \cdot \mathbb{C}$. Hence $\beta \circ \gamma = \lambda \cdot \mathrm{id}_L$ for some $\lambda \in \mathbb{C}$. If $\lambda \neq 0$, $\beta \circ (\frac{1}{\lambda}\gamma) = \mathrm{id}_L$ and $\frac{1}{\lambda}\gamma : L \to E$ would split the sequence but we know the sequence (2.10) is nonsplit, hence $\beta \circ \gamma = 0$. It follows from this that $\beta \circ \phi = \beta \circ \gamma \circ \beta = 0$. In other words, the image of ϕ is contained in the line subbundle $\mathcal{O} \subset E$, so that ϕ is induced by an element of $\mathrm{Hom}(L, \mathcal{O})$:

$$\phi : E \to L \to \mathcal{O} \to E.$$

However, we have $\mathrm{Hom}(L, \mathcal{O}) \cong H^0(L^*) = 0$. Hence, $\phi = 0$ if $H^0(\phi) = 0$.

If, on the other hand, $H^0(\phi) \neq 0$, then because $h^0(E) = 1, H^0(\phi)$ is multiplication by a constant $c \in \mathbb{C}$. Then by considering $\phi - c \cdot \mathrm{id}_E$ we reduce to the previous case and this shows that $\operatorname{End} E = \mathbb{C}$. Hence, E is simple and thus E is indecomposable. We have therefore proved the existence part of the proposition, and it remains to show uniqueness.

Fixing L of degree 1 with $\det E \cong L$, we have $H^0(E) \neq 0$ by Riemann-Roch, i.e. $\operatorname{Hom}(\mathcal{O}, E) \neq 0$. Hence E contains $\mathcal{O}$ as a subsheaf. By applying saturation (Remark 2.1.16) to this subsheaf we get the following diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, hook] & E \arrow[r, "g"] \arrow[d, equal] & \mathcal{F} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & E' \arrow[r] & E \arrow[r] & \mathcal{G} \arrow[r] & 0 \end{tikzcd} \tag{2.15}$$

in which $E' \subset E$ is a line subbundle and so $\mathcal{G}$ is locally free. From Lemma 2.1.31 we get $\deg(E') \leq 0$. Since $\mathcal{O} \hookrightarrow E'$, we have $H^0(E') \neq 0$. This implies by Lemma 2.1.30 that $\deg(E') = 0$ and $E' \cong \mathcal{O}$. The second row of (2.15) is therefore of the following form

$$0 \to \mathcal{O} \to E \to \mathcal{G} \to 0$$

Because $\det E \cong \mathcal{G}$, we have $\mathcal{G} = L$. As $H^1(L) = 1$, up to isomorphism E is uniquely determined by L and so E is precisely the bundle constructed above. □

Proposition 2.3.5. *On a smooth curve, C, of genus 1 every indecomposable rank 2 vector bundle of even degree is an extension of the form*

$$0 \longrightarrow M \longrightarrow E \longrightarrow M \longrightarrow 0$$

for some line bundle M on E.

Proof. ([Mu] Proposition 10.48) Let $\deg E = 2k$. If $M_1 \in \mathrm{Pic}^{-k}(C)$, i.e. M_1 is a line bundle of degree $-k$, then $E \otimes M_1$ is of degree 0. In other words, $E \otimes M_1 \in \mathcal{E}(r, 0)$. By Theorem 2.3.2, we know that there exists $M_2 \in \mathrm{Pic}^0(C)$

such that $E \otimes M_1 \cong E_2 \otimes M_2$, where E_2 is the so-called Atiyah bundle from Theorem 2.3.2 and sits in a nonsplit exact sequence as follows

$$0 \to \mathcal{O}_C \to E_2 \to \mathcal{O}_C \to 0.$$

If $M := M_2 \otimes M_1^*$, we obtain $E \cong E_2 \otimes M$ and tensoring this sequence by M, gives a nonsplit short exact sequence

$$0 \to M \to E \to M \to 0$$

$\square$

Proposition 2.3.6. *On a smooth elliptic curve, C, there is a bijection between C and $\mathrm{Pic}^n(C)$ for all $n \in \mathbb{Z}$, where $\mathrm{Pic}^n(C)$ denotes the set of degree n line bundles on C.*

Proof. ([H] Example 1.3.7) Let us first show that there is a bijection between C and $\mathrm{Pic}^0(C)$. Fix a point $P_0 \in C$, define a map

$$C \to \mathrm{Pic}^0(C)$$

given by

$$P \mapsto \mathcal{O}(P - P_0) \text{ with } P \in C.$$

Now if P and Q are distinct points on C and $\mathcal{O}(P - P_0) \cong \mathcal{O}(Q - P_0)$, we get $\mathcal{O}(P - Q) \cong \mathcal{O}$. We know from Lemma 1.2.16 that $P - Q = \mathrm{div}(f)$ for some meromorphic function f on C. This means that f has exactly one pole of order 1 at Q and one zero of order 1 at P and no other poles or zeros. Therefore we can define an isomorphism $g : C \to \mathbb{P}^1$, given by $x \mapsto (f(x) : 1)$ if $x \neq Q$ and $Q \mapsto (1 : 0)$. Since f is a meromorphic function with exactly one pole and is holomorphic elsewhere, g is an isomorphic. However, C has genus 1 and $\mathbb{P}^1$ has genus 0, therefore they could not be isomorphic so we have a contradiction. Hence $\mathcal{O}(P - Q) \ncong \mathcal{O}$, unless $P = Q$. So we have

$$C \hookrightarrow \mathrm{Pic}^0(C)$$

Now let us start with a line bundle, $L \in \mathrm{Pic}^0(C)$. Tensor this with $\mathcal{O}(P_0)$ for $P_0 \in C$, fixed. Since $\deg(L \otimes \mathcal{O}(P_0)) = 1$, by Lemma 2.1.30(b) we have that $h^1(L \otimes \mathcal{O}(P_0)) = 0$ and by Riemann-Roch we know then that $h^0(L \otimes \mathcal{O}(P_0)) = 1$. Hence we know that there exists $s \in H^0(L \otimes \mathcal{O}(P_0)), s \neq 0$ with

$$\mathrm{div}(s) = \sum_P \mathrm{ord}_P(s) \cdot P$$

Now, since s is a holomorphic section, $\mathrm{ord}_P(s) \geq 0$ for all $P \in C$. Hence $\sum \mathrm{ord}_p(s) = 1$ and this in turn implies that $\mathrm{div}(s) = P$. Hence $L \cong \mathcal{O}(P)$ and so we have a bijection between C and $\mathrm{Pic}^0(C)$.

We also have a bijection between $\mathrm{Pic}^0(C)$ and $\mathrm{Pic}^n(C)$ given by

$$\mathrm{Pic}^0(C) \to \mathrm{Pic}^n(C)$$

$$L \mapsto L \otimes \mathcal{O}(nP_0)$$

whose inverse is

$$\mathrm{Pic}^n(C) \to \mathrm{Pic}^0(C)$$

$$M \mapsto M \otimes \mathcal{O}(-nP_0)$$

where $L \in \mathrm{Pic}^0(C)$ and $M \in \mathrm{Pic}^n(C)$. □

Theorem 2.3.7. *For each integer n, there is a one-to-one correspondence between the set of isomorphism classes of indecomposable vector bundles of rank 2 and degree n on the elliptic curve C, and the set of points on C.*

Proof. ([H] Corollary 2.16) Let E be an indecomposable rank 2 vector bundle of degree n on C. If n is odd, from Proposition 2.3.4, we know that there is a unique indecomposable rank 2 vector bundle E of degree n, with $\det E \cong L$. So we then use the bijection described in Proposition 2.3.6 to obtain the result.

If $n = 2k$ is even, we pick a line bundle, M, of degree $-k$ so that $E \otimes M$ is of degree 0. From Theorem 2.3.2, we know there exists, L, a unique line bundle of degree zero, such that $E_2 \otimes L \cong E \otimes M$, where E_2 is the unique nontrivial extenstion of $\mathcal{O}_C$ by $\mathcal{O}_C$. Since the line bundles of degree 0 are in bijection with the points of C we obtain the result. □

2.4 Stability

The motivation for stability comes from Geometric Invariant Theory (G.I.T), which was pioneered by David Mumford. This theory is used in constructing moduli spaces. Using G.I.T, we need 'stable' vector bundles in order to construct moduli spaces. For example, using stable bundles it is possible to construct the moduli space of (stable) vector bundles of rank 2 and degree d on C. In general, a moduli space is an algebraic variety which parametrises the set of equivalence classes of some objects. To begin with we will give an example of a moduli space.

Example 2.4.1. Let C be an elliptic curve. There is a line bundle on $C \times \mathrm{Pic}^0(C) \cong C \times C$ called the *Poincaré bundle*, denoted $\mathcal{P}$. It has the property that $\mathcal{P}|_{C\times\{P\}} \cong \mathcal{O}(P - P_0)$ for some point P_0 on C and for all $P \in C$.

Let us now show how to construct the Poincaré bundle. Consider the subset $\Delta = \{(P,Q)|P = Q\} \subset C \times C$ called the *diagonal*. According to Remark 1.2.2, Δ is a divisor on $C \times C$ since locally it is the zero locus of a single equation. According Remark 1.2.19, we can construct the associated line bundle $\mathcal{O}(\Delta)$. Let pr_1 and pr_2 denote the projection of $C \times \mathrm{Pic}^0(C) \cong C \times C$ onto the first and second factor. Then $\mathrm{pr}_1^{-1}(P_0) = \{P_0\} \times C \subset C \times C$ and $\mathrm{pr}_2^{-1}(P_0) = C \times \{P_0\}$ are divisors on $C \times C$. The line bundle $\mathcal{P} := \mathcal{O}_{C\times C}(\Delta - \{P_0\} \times C - C \times \{P_0\})$ is the (normalised) Poincaré bundle. It satisfies

$$\begin{aligned}\mathcal{P}|_{C\times\{P\}} &= \mathcal{O}_{C\times\{P\}}((P,P) - (P_0,P))\\ &= \mathcal{O}_C(P - P_0)\end{aligned}$$

and

$$\mathcal{P}|_{\{P_0\}\times C} = \mathcal{O}_C(P_0 - P_0) = \mathcal{O}_C.$$

This is an example of a moduli space, $\mathrm{Pic}^0(C)$, and it's universal bundle, $\mathcal{P}$.

Let us now introduce the notion of a stable vector bundle.

Definition 2.4.2. A vector bundle, E on a curve, is *stable* (resp. *semi-stable*) if every nonzero vector subbundle $F \subset E$ satisfies

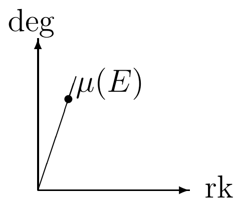
$$\frac{\deg F}{\operatorname{rk} F} < \frac{\deg E}{\operatorname{rk} E} \quad (\text{resp. } \leq).$$

(Or equivalently, we can also say that a vector bundle E is stable (resp. semi-stable) if $\frac{\deg G}{\operatorname{rk} G} > \frac{\deg E}{\operatorname{rk} E}$ (resp. $\geq$) for every nonzero quotient G of E).

From this definition we can see that a vector bundle E of $\operatorname{rk} 2$ is stable (resp. semi-stable) if every line subbundle $F \subset E$ satifies

$$\deg F < \frac{1}{2} \deg E \quad (\text{resp. } \leq).$$

We call the rational number $\mu(E) := \frac{\deg E}{\operatorname{rk} E}$ the *slope* of E. The picture below illustrates the reason for this name.



Lemma 2.4.3. *Let E be a vector bundle of rank 2. If $\deg E$ is odd, then stability and semi-stabilty are equivalent.*

Proof. ([Mu] Remark 10.21) Clearly if E is stable, then E is semi-stable. For the other direction, we assume E is semi-stable with degree n. Now let $F \subset E$ be a nonzero subbundle of E, so $\deg F \leq \frac{n}{2}$. Since $\deg F$ is an integer, $\deg F \neq \frac{n}{2}$ as n is odd. Hence $\deg F < \frac{n}{2}$, i.e. E is stable. $\square$

Lemma 2.4.4. *If E_1 and E_2 are semi-stable vector bundles, and $\mu(E_1) > \mu(E_2)$, then* $\operatorname{Hom}(E_1, E_2) = 0$

Proof. ([Ar] Section 1) Let $f : E_1 \to E_2$ be a morphism, and let $F \subset E_2$ be it's image, which is again a vector bundle by Remark 1.3.22. Since E_2 is semi-stable, if $F \neq 0$, then $\frac{\deg F}{\operatorname{rk} F} \leq \frac{\deg E_2}{\operatorname{rk} E_2}$. But E_1 is semi-stable and F is a quotient of E_1, and therefore $\frac{\deg E_1}{\operatorname{rk} E_1} \leq \frac{\deg F}{\operatorname{rk} F}$, a contradiction unless $F = 0$. $\square$

2.4.1 Harder-Narasimhan filtrations

Each vector bundle admits a canonical increasing filtration called the Harder-Narasimhan filtration whose successive quotients are semi-stable. This allows us to describe bundles which are not semi-stable in terms of semi-stable bundles. In order to prove that such a filtration exists we first need the following lemma.

Lemma 2.4.5. *(a) Let $d, d', r, r' \in \mathbb{Z}$ with $r, r' > 0$.*

(i) If $\frac{d}{r} > \frac{d'}{r'}$, then $\frac{d}{r} > \frac{d+d'}{r+r'} > \frac{d'}{r'}$.

(ii) If $\frac{d}{r} = \frac{d+d'}{r+r'}$ or $\frac{d'}{r'} = \frac{d+d'}{r+r'}$ then $\frac{d}{r} = \frac{d'}{r'}$.

(b) Let $0 \to E' \to E \to E'' \to 0$ be a short exact sequence of nonzero vector bundles on X.

(i) If $\lambda \in \mathbb{R}$ such that $\mu(E') \leq \lambda$ and $\mu(E'') \leq \lambda$, then $\mu(E) \leq \lambda$.

(ii) If one of the assumed inequalities in (i) is strict, then $\mu(E) < \lambda$.

(iii) If $\mu(E') = \mu(E)$ or $\mu(E) = \mu(E'')$ then $\mu(E') = \mu(E'')$.

(c) If

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_n = E$$

is a filtration by subbundles of E such that $\mu(E_i/E_{i-1}) \leq \lambda$ for all $i = 1, \ldots, n$

(i) then $\mu(E_i) \leq \lambda$ for all $i = 1, \ldots, n$. In particular, $\mu(E) \leq \lambda$.

(ii) If, for at least one i, we have $\mu(E_i/E_{i-1}) < \lambda$, then $\mu(E) < \lambda$.

Proof. (a) The proof of this is a simple calculation.

(b) Because $\mathrm{rk}(E) = \mathrm{rk}(E') + \mathrm{rk}(E'')$ and $\deg(E) = \deg(E') + \deg(E'')$, this follows immediately from (a).

(c) This follows from (b) using exact sequences

$$0 \to E_{i-1} \to E_i \to E_i/E_{i-1} \to 0$$

for all $i = 2, \ldots n$. □

We are now ready to prove the existence and uniqueness of the Harder-Narasimhan filration for each vector bundle on a curve.

Proposition 2.4.6. *Let E be a vector bundle on a curve C. Then E has an increasing filtration by vector subbundles*

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_k = E$$

where the quotients $\mathrm{gr}_i = E_i/E_{i-1}$ satisfy the following conditions:

1. *the quotient gr_i is semi-stable;*

2. *$\mu(\mathrm{gr}_i) > \mu(\mathrm{gr}_{i+1})$ for $i = 1, \cdots, k-1$.*

Proof. ([P01] Proposition 5.4.2) If E is already semi-stable then the result is trivial. Assume, therefore that E is not semi-stable. We will prove this by induction on the rank of E. If $\mathrm{rk}(E) = 1$, then the result is trivial as all line bundles are automatically stable. Now assume $\mathrm{rk}(E) \geq 2$. We know, from Lemma 2.1.32, that the degree of all subbundles of E is bounded above. On the other hand, subbundles can only have ranks $1, 2, \ldots, \mathrm{rk}(E) - 1$, hence the slope of the subbundles of E is bounded above. Let E_1 be a subbundle of maximal rank among all the subbundles of maximal slope. Then E_1 is semi-stable because it has maximal slope. Let $E' = E/E_1$, then we have the following short exact sequence:

$$0 \to E_1 \to E \to E' \to 0$$

where $\mathrm{rk}(E') < \mathrm{rk}(E)$.

By inductive assumption E' has an increasing filtration satisfying the conditions of the proposition, i.e.

$$0 \subset F_2 \subset F_3 \subset \cdots \subset F_k = E'$$

with $\mu(F_2) > \mu(F_3/F_2) > \cdots > \mu(F_k/F_{k-1})$ and F_j/F_{j-1} is semi-stable for $2 \leq j \leq k$. In particular F_2 is semi-stable.

Let $E_j \subset E$ be the preimage of $F_j \subset E'$ under $E \to E'$. This way we obtain a filtration $0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_k = E$ and commutative diagrams with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & E_1 \arrow[r] & E_{j+1} \arrow[r] & F_{j+1} \arrow[r] & 0 \\
0 \arrow[r] & E_1 \arrow[r] \arrow[u, equal] & E_j \arrow[r] \arrow[u, hook] & F_j \arrow[r] \arrow[u, hook] & 0.
\end{tikzcd}$$

Hence $E_{j+1}/E_j \cong F_{j+1}/F_j$ are semi-stable for all $j = 1, \ldots, k-1$.

Now we need to prove $\mu(F_2) < \mu(E_1)$, in order to show condition 2 holds. Since E_1 has maximal slope, $\mu(E_2) \leq \mu(E_1)$. Moreover, since E_1 has maximal rank among the subbundles with slope $\mu(E_1), \mu(E_2) < \mu(E_1)$. From the diagram above we know that $\deg(F_2) = \deg(E_2) - \deg(E_1)$ and $\mathrm{rk}(F_2) = \mathrm{rk}(E_2) - \mathrm{rk}(E_1)$. So we know $\mu(F_2) = \frac{\deg(E_2)-\deg(E_1)}{\mathrm{rk}(E_2)-\mathrm{rk}(E_1)}$. We can also write this as $\mu(F_2) = \frac{\mathrm{rk}(E_2)\mu(E_2)-\mathrm{rk}(E_1)\mu(E_1)}{\mathrm{rk}(E_2)-\mathrm{rk}(E_1)}$. Then we have

$$\frac{\mathrm{rk}(E_2)\mu(E_2) - \mathrm{rk}(E_1)\mu(E_1)}{\mathrm{rk}(E_2) - \mathrm{rk}(E_1)} < \frac{\mathrm{rk}(E_2)\mu(E_1) - \mathrm{rk}(E_1)\mu(E_1)}{\mathrm{rk}(E_2) - \mathrm{rk}(E_1)}$$

i.e. $\mu(F_2) < \mu(E_1)$. Now since $E_2/E_1 = F_2$, we have $\mu(E_1) > \mu(E_2/E_1)$. $\square$

Lemma 2.4.7. *If*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E$$

is a filtration of E satsifying the conditions of Proposition 2.4.6 and $E' \subset E$ is a nontrivial subbundle of E then $\mu(E') \leq \mu(E_1)$ and if $\mu(E') = \mu(E_1)$, then $E' \subset E_1$.

Proof. We define a filtration of E' by $E'_i := E' \cap E_i$ for all $i = 1, \ldots, n$. Because $E'_i = E_i \cap E'_{i+1}$ we obtain $E'_{i+1}/E'_i \subset E_{i+1}/E_i$ for $i = 1, \ldots, n-1$. Now since E_{i+1}/E_i is semi-stable, we have either $\mu(E'_{i+1}/E'_i) \leq \mu(E_{i+1}/E_i)$ or $E'_{i+1} = E'_i$. Because $\mu(E_{i+1}/E_i) \leq \mu(E_1)$ for $i = 1, 2, \ldots, n-1$, we obtain from Lemma 2.4.5 (c) that $\mu(E') \leq \mu(E_1)$. Now if there exists $i \geq 1$ with $E'_{i+1} \neq E'_i$ then $\mu(E'_{i+1}/E'_i) \leq \mu(E_{i+1}/E_i) < \mu(E_1)$. Hence by Lemma 2.4.5 (c) again, $\mu(E') < \mu(E_1)$. Hence, if $\mu(E') = \mu(E_1)$ we must have $E'_{i+1} = E'_i$ for $i = 1, 2, \ldots, n-1$, i.e. $E' \subset E_1$. $\square$

Proposition 2.4.8. *This filtration of Proposition 2.4.6 is unique.*

Proof. ([P01] Proposition 5.4.2) Assume $(E_i)_{i=1,\ldots,n}$ and $(F_j)_{j=1,\ldots,m}$ are two filtrations of E satisfying the conditions of Proposition 2.4.6 above. Now using the notation of Lemma 2.4.7 if we let $E' := F_1$ we get $\mu(F_1) \leq \mu(E_1)$. Similarly if we allow $E' := E_1$, we get $\mu(E_1) \leq \mu(F_1)$. Clearly then, $\mu(F_1) = \mu(E_1)$.

Lemma 2.4.7 again implies $E_1 \subset F_1$ and $F_1 \subset E_1$, hence $E_1 = F_1$. Using E/E_1 and F/F_1 we can proceed by induction as in the proof of Proposition 2.4.6 to conclude that the filtration is unique. □

The filtration of Proposition 2.4.6 is called the *Harder-Narasimhan filtration* of E.

Chapter 3

Bridgeland stability

In this chapter we outline some of the results of Bridgeland ([Br01] and [Br02]). In Section 2.4 we have seen the definition of stability for a vector bundle on a curve C. The notion of stability was generalised by Rudakov ([R]), to give the notion of stability on an abelian category. For the rest of this chapter $\mathcal{A}$ will denote an abelian category, unless otherwise specified. To begin with, we need to introduce the notion of a stability function.

3.1 Stability conditions

Definition 3.1.1. Let M be the free abelian group and generated by isomorphism classes of objects in $\mathcal{A}$, (for all objects $E \in \mathcal{A}$, we also denote its isomorphism class by E). In this free abelian group, let Γ be the subgroup generated by all elements

$$E - F - G$$

for which there exists a short exact sequence $0 \to F \to E \to G \to 0$ in $\mathcal{A}$. The quotient group M/Γ is called the *Grothendieck group*, $K(\mathcal{A})$ and an element of the Grothendieck group will be denoted $[E]$.

Example 3.1.2. Let $\mathcal{A} = \mathbf{Coh}(C)$ be the abelian category of coherent sheaves on C, a smooth projective curve. The Grothendieck group,

$K(\mathbf{Coh}(C)) = K(C)$ is then described by the following isomorphism (see [M] Remark 10.9)

$$K(C) \xrightarrow{\sim} \mathbb{Z} \oplus \mathrm{Pic}(C), \quad [\mathcal{F}] \mapsto (\mathrm{rk}(\mathcal{F}), \det \mathcal{F}).$$

Definition 3.1.3. A *stability function* on $\mathcal{A}$ is a group homomorphism $Z : K(\mathcal{A}) \to \mathbb{C}$ such that for all nonzero $E \in \mathcal{A}$ the complex number $Z(E)$ lies in the strict upper half-plane $H = \{r \cdot \exp(i\pi\phi) | r > 0 \text{ and } 0 < \phi \leq 1\} \subset \mathbb{C}$.

Using a stability function, we can define the phase of an object in $\mathcal{A}$ which plays the role that the slope played for vector bundles on a curve.

Definition 3.1.4. Given a stability function $Z : K(\mathcal{A}) \to \mathbb{C}$, the *phase* of a nonzero object $[E] \in K(\mathcal{A})$ is defined to be

$$\phi(E) = \frac{1}{\pi} \arg Z(E) \in (0, 1].$$

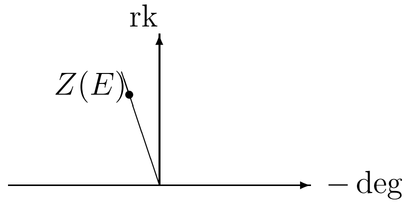
The function ϕ allows us to order the nonzero objects of $\mathcal{A}$ and thus leads to a notion of stability for objects of $\mathcal{A}$.

Definition 3.1.5. Let $Z : K(\mathcal{A}) \to \mathbb{C}$ be a stability function on an abelian category $\mathcal{A}$. A nonzero object $E \in \mathcal{A}$ is called *semi-stable* (with respect to Z) if $\phi(F) \leq \phi(E)$ for every nonzero subobject $F \subset E$. Equivalently, E is semi-stable if $\phi(G) \geq \phi(E)$ for every nonzero quotient G of E.

Example 3.1.6. Let C be a smooth projective curve and let $\mathbf{Coh}(C)$ be the abelian category of coherent sheaves on C. Consider the stability function

$$Z(E) = -\deg(E) + i\,\mathrm{rk}(E)$$

on $\mathbf{Coh}(C)$. An object $E \in \mathbf{Coh}(C)$ is semi-stable with respect to Z if and only if it is semi-stable with respect to Definition 2.4.2.



The following is the Harder-Narasimhan property, which was outlined for vector bundles on curves in Section 2.4.1.

Definition 3.1.7. Let $Z : K(\mathcal{A}) \to \mathbb{C}$ be a stability function on an abelian category $\mathcal{A}$. A *Harder-Narasimhan filtration* of a nonzero object $E \in \mathcal{A}$ is a finite chain of subobjects

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

whose factors $F_j = E_j/E_{j-1}$ are semi-stable objects of $\mathcal{A}$ and satisfy

$$\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n).$$

The stability function Z is said to have the *Harder-Narasimhan property* if every nonzero object of $\mathcal{A}$ has a Harder-Narasimhan filtration.

Not every stability function has the Harder-Narasimhan property. The following proposition follows from a result of Rudakov ([R]).

Proposition 3.1.8. *Suppose $\mathcal{A}$ is an abelian category with a stability function $Z : K(\mathcal{A}) \to \mathbb{C}$. The stability function, Z, has the Harder-Narasimhan property if the following two conditions are satisfied:*

(a) There are no infinite sequences of monomorphisms in $\mathcal{A}$

$$\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1$$

with $\phi(E_{j+1}) > \phi(E_j)$ for all j.

(b) There are no infinite sequences of epimorphisms in $\mathcal{A}$

$$E_1 \twoheadrightarrow E_2 \twoheadrightarrow \cdots \twoheadrightarrow E_j \twoheadrightarrow E_{j+1} \twoheadrightarrow \cdots$$

with $\phi(E_j) > \phi(E_{j+1})$ for all j.

Proof. See [Br01] Proposition 2.4 □

Let $\mathcal{A}$ be an abelian category and let $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$ be the bounded derived category. We now need to define the Grothendieck group of $\mathcal{D}$ as follows:

Definition 3.1.9. Let M be the free abelian group generated by isomorphism classes of objects in $\mathcal{D}$. In this free abelian group, let Γ be the subgroup generated by all elements

$$X^\bullet - Y^\bullet - Z^\bullet$$

for which there exists a distinguished triangle $Y^\bullet \to X^\bullet \to Z^\bullet \to Y[1]^\bullet$ in $\mathcal{D}$. The quotient group M/Γ is called the *Grothendieck group*, $K(\mathcal{D})$.

Proposition 3.1.10. *There is an isomorphism $K(\mathcal{A}) \to K(\mathcal{D})$.*

Proof. [Gr], Section 4. $\square$

Let us now introduce the notion of a stability condition on the bounded derived category, $\mathcal{D}$.

Definition 3.1.11. A *stability condition* $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ consists of a group homomorphism $Z : K(\mathcal{D}) \to \mathbb{C}$ called the *central charge*, and a *slicing* $\mathcal{P}$ of $\mathcal{D}$, which by definition consists of full additive subcategories $\mathcal{P}(\phi) \subset \mathcal{D}$ for each $\phi \in \mathbb{R}$, satisfying the following axioms:

(a) If $0 \neq X^\bullet \in \mathcal{P}(\phi)$, then $Z(X) = m(X)\exp(i\pi\phi)$ for some $m(X) \in \mathbb{R}_{>0}$.

(b) For all $\phi \in \mathbb{R}, \mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$.

(c) If $X_j^\bullet \in \mathcal{P}(\phi_j)$ for $j = 1, 2$ and $\phi_1 > \phi_2$, then $\mathrm{Hom}_{\mathcal{D}}(X_1^\bullet, X_2^\bullet) = 0$.

(d) For each nonzero object $X^\bullet \in \mathcal{D}$ there exists a finite collection of distinguished triangles

$$X_{j-1}^\bullet \to X_j^\bullet \to A_j^\bullet \to X_{j-1}[1]^\bullet \quad (1 \leq j \leq n)$$

with $X_0^\bullet = 0, X_n^\bullet = X$, and $A_j^\bullet \in \mathcal{P}(\phi_j)$ for all j, such that

$$\phi_1 > \phi_2 > \cdots > \phi_n.$$

The nonzero objects of $\mathcal{P}(\phi)$ are said to be *semi-stable* in σ of phase ϕ, and the simple objects (i.e. those which have no nonzero proper subobjects) of $\mathcal{P}(\phi)$ are said to be *stable*.

For any interval $I \subset \mathbb{R}$, define $\mathcal{P}(I)$ to be the extension-closed subcategory of $\mathcal{D}$ (A full subcategory $\mathcal{A}$ of $\mathcal{D}$ is called extension-closed if whenever $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$ is a distinguished triangle in $\mathcal{D}$, with $A \in \mathcal{A}$ and $C \in \mathcal{A}$, then $B \in \mathcal{A}$ also) generated by the subcatgories $\mathcal{P}(\phi)$ for $\phi \in I$.

Now let $\mathcal{A} = \mathbf{Coh}(C)$, where C is a smooth projective curve and let $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$ be its bounded derived category. In this case we have the notion of a numerical Grothendieck group given by the following definition:

Definition 3.1.12. We define a bilinear form on $K(\mathcal{D})$, known as the *Euler form*, by

$$\chi(E^\bullet, F^\bullet) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}}(E^\bullet, F[i]^\bullet),$$

and a free abelian group $\mathcal{N}(\mathcal{D}) = K(\mathcal{D})/K(\mathcal{D})^\perp$, called the *numerical Grothendieck group* of $\mathcal{D}$, where

$$K(\mathcal{D})^\perp = \{E^\bullet \in K(\mathcal{D}) | \chi(E^\bullet, F^\bullet) = 0 \text{ for all } F^\bullet \in K(\mathcal{D})\}.$$

The following lemma ([H] Exercise II.6.11) about coherent sheaves will be useful in proofs later on.

Lemma 3.1.13. *Let $\mathcal{F}$ be a coherent sheaf on a smooth projective curve, C. Then there exist locally free sheaves (i.e. holomorphic vector bundles) $\mathscr{E}_0$ and $\mathscr{E}_1$ and an exact seqeunce*

$$0 \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathcal{F} \to 0.$$

We call the exact sequence above a locally free resolution of $\mathcal{F}$.

Remark 3.1.14. If $\mathcal{F}$ is a coherent sheaf on C and

$$0 \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathcal{F} \to 0$$

is a locally free resolution of $\mathcal{F}$, then we define the degree, rank and determinant of $\mathcal{F}$ as follows:

$$\deg(\mathcal{F}) := \deg(\mathscr{E}_0) - \deg(\mathscr{E}_1)$$

$$\operatorname{rk}(\mathcal{F}) := \operatorname{rk}(\mathscr{E}_0) - \operatorname{rk}(\mathscr{E}_1)$$

$$\det(\mathcal{F}) = \det(\mathcal{E}_0) \otimes \det(\mathcal{E}_1)^*.$$

Note that this coincides with out definition of degree, rank and determinant of a vector bundle if $\mathcal{F}$ were locally free (See Remark 2.1.11 and page 62).

Example 3.1.15. Let $\mathbf{Coh}(C)$ be the category of coherent sheaves on an elliptic curve, C. Let $\mathcal{D} = \mathcal{D}^b(\mathbf{Coh}(C))$ be its bounded derived category. Set $K(C) = K(\mathcal{D}^b(\mathbf{Coh}(C))$. We want to compute the numerical Grothendieck group $\mathcal{N}(C) = K(C)/K(C)^{\perp}$. Let $\mathcal{E}^\bullet, \mathcal{F}^\bullet \in \mathcal{D}^b(\mathbf{Coh}(C))$. The Euler form

$$\chi(\mathcal{E}^\bullet, \mathcal{F}^\bullet) = \sum_i (-1)^i \dim_{\mathbb{C}} \operatorname{Hom}_{\mathcal{D}}(\mathcal{E}^\bullet, \mathcal{F}[i]^\bullet),$$

can also be written as (from Remark 1.4.68)

$$\chi(\mathcal{E}, \mathcal{F}) = \sum_i (-1)^i \dim_{\mathbb{C}} \operatorname{Ext}^i(\mathcal{E}, \mathcal{F}).$$

Assume for the moment that one of $\mathcal{E}$ or $\mathcal{F}$ is locally free. Since C is a curve we then get

$$\begin{aligned}\chi(\mathcal{E}, \mathcal{F}) &= \dim \operatorname{Ext}^0(\mathcal{E}, \mathcal{F}) - \dim \operatorname{Ext}^1(\mathcal{E}, \mathcal{F}) \\ &= h^0(\mathcal{E}^* \otimes \mathcal{F}) - h^1(\mathcal{E}^* \otimes \mathcal{F}),\end{aligned}$$

where the last equality uses Proposition 2.1.18. Using the Riemann-Roch formula we get

$$\begin{aligned}\chi(\mathcal{E}, \mathcal{F}) &= \deg(\mathcal{E}^* \otimes \mathcal{F}) \\ &= -\deg(\mathcal{E}) \operatorname{rk}(\mathcal{F}) + \operatorname{rk}(\mathcal{E}) \deg(\mathcal{F}).\end{aligned}$$

This also works if neither of $\mathcal{E}$ or $\mathcal{F}$ are locally free. If $\mathcal{E}$ is any coherent sheaf on a smooth projective curve, there exists a locally free resolution of $\mathcal{E}$ (Lemma 3.1.13) as follows:

$$0 \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathcal{E} \to 0.$$

Now $\chi(-, \mathcal{F})$ is an additive function on short exact sequences so we have

$$\chi(\mathcal{E}, \mathcal{F}) = \chi(\mathscr{E}_0, \mathcal{F}) - \chi(\mathscr{E}_1, \mathcal{F})$$

Let $d_{\mathcal{F}} = \deg(\mathcal{F})$, $r_{\mathcal{F}} = \mathrm{rk}(\mathcal{F})$ and let $d_i = \deg(\mathscr{E}_i)$ and $r_i = \mathrm{rk}(\mathscr{E}_i)$ for $i = 1, 2$. Using Proposition 2.1.18 and Riemann-Roch as above, we then get

$$\begin{aligned} \chi(\mathcal{E}, \mathcal{F}) &= r_0 d_{\mathcal{F}} - r_{\mathcal{F}} d_0 - (r_1 d_{\mathcal{F}} - r_{\mathcal{F}} d_1) \\ &= (r_0 - r_1) d_{\mathcal{F}} - r_{\mathcal{F}}(d_0 - d_1) \\ &= \mathrm{rk}(\mathcal{E}) \deg(\mathcal{F}) - \mathrm{rk}(\mathcal{F}) \deg(\mathcal{E}). \end{aligned}$$

Now by definition of $K(C)^{\perp}$ we know that $[\mathcal{E}] \in K(C)^{\perp}$ if and only if $\mathrm{rk}(\mathcal{E}) = \deg(\mathcal{E}) = 0$. Now consider the short exact sequence

$$0 \to K(C)^{\perp} \to K(C) \to \mathbb{Z}^2 \to 0$$

where $K(C) \to \mathbb{Z}^2$ is given by $[\mathcal{F}] \mapsto (\mathrm{rk}(\mathcal{F}), \deg(\mathcal{F}))$ and $K(C)^{\perp}$ is the kernel of $K(C) \to \mathbb{Z}^2$. Hence we get $K(C)/K(C)^{\perp} \cong \mathbb{Z}^2$, i.e. $\mathcal{N}(C) = \mathbb{Z}^2$.

We also have the notion of a numerical stability condition as follows:

Definition 3.1.16. A stability condition $(Z, \mathcal{P})$ on $\mathcal{D}$, the bounded derived category of coherent sheaves on a smooth projective curve, is said to be *numerical* if the central charge $Z : K(\mathcal{D}) \to \mathbb{C}$ factors through the quotient group $\mathcal{N}(\mathcal{D}) = K(\mathcal{D})/K(\mathcal{D})^{\perp}$, i.e. if there exists a homomorphism of groups $f : \mathcal{N}(\mathcal{D}) \to \mathbb{C}$ such that $Z = f \circ g$ in the following diagram, where g is the quotient map

$$\begin{tikzcd} K(\mathcal{D})^{\perp} \arrow[r] & K(\mathcal{D}) \arrow[r, two heads, "g"] \arrow[d, "Z"] & \mathcal{N}(\mathcal{D}) \arrow[dl, dashed, "f"] \\ & \mathbb{C} & \end{tikzcd} .$$

3.2 t-structures

The notion of a t-structure was introduced by A. Beilinson, J. Bernstein and P. Deligne in [BBD]. T-structures are the tool which allows one to see the different abelian categories embedded in a given triangulated category (in our case the bounded derived category $\mathcal{D}$, of an abelian category $\mathcal{A}$. See Remark 1.4.57). Let us now give the definition of a t-structure.

Definition 3.2.1. A *t-structure* on $\mathcal{D}$ is a pair of strictly full subcategories (See Definition 1.4.24) $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ such that, if we let $\mathcal{D}^{\leq n} = \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} = \mathcal{D}^{\geq 0}[-n]$ for every $n \in \mathbb{Z}$, then

(a) $\mathcal{D}^{\leq 0} \subseteq \mathcal{D}^{\leq 1}, \mathcal{D}^{\geq 0} \supseteq \mathcal{D}^{\geq 1}$.

(b) $\mathrm{Hom}_{\mathcal{D}}(X^\bullet, Y^\bullet) = 0$ for every $X^\bullet \in \mathcal{D}^{\leq 0}$ and $Y^\bullet \in \mathcal{D}^{\geq 1}$.

(c) For any $X^\bullet \in \mathcal{D}$ there exists a distinguished triangle

$$Y^\bullet \to X^\bullet \to Z^\bullet \to Y[1]^\bullet$$

such that $Y^\bullet \in \mathcal{D}^{\leq 0}$ and $Z^\bullet \in \mathcal{D}^{\geq 1}$.

Definition 3.2.2. The *heart* of a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ is the full subcategory $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} \subset \mathcal{D}$. The *standard t-structure* on $\mathcal{D}$ is defined by

$$\mathcal{D}^{\leq 0} = \{X^\bullet \in \mathcal{D} | H^p(X^\bullet) = 0 \; \forall p > 0\}, \mathcal{D}^{\geq 0} = \{X^\bullet \in \mathcal{D} | H^p(X^\bullet) = 0 \; \forall p < 0\}.$$

The heart of the standard t-structure is $\mathcal{A}$.

Definition 3.2.3. A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathcal{D}$ is called *bounded* if

$$\bigcup_{n \in \mathbb{Z}} \mathcal{D}^{\leq n} = \mathcal{D} = \bigcup_{n \in \mathbb{Z}} \mathcal{D}^{\geq n}$$

Example 3.2.4. The standard t-structure is bounded. Clearly $\bigcup_{n \in \mathbb{Z}} \mathcal{D}^{\leq n} \subset \mathcal{D}$. Recall that if $X^\bullet$ is an object of $\mathcal{D}$, then there exists integers m and M such that $H^n(X^\bullet) = 0$ for all $n < m$ and $n > M$. Hence, $X^\bullet \in \mathcal{D}^{\geq m} \cap \mathcal{D}^{\leq M}$ and so $\mathcal{D} \subset \bigcup_{n \in \mathbb{Z}} \mathcal{D}^{\leq n}$. Similarly the equality on the right can be shown.

The main result that we can now state about stability conditions is the following:

Proposition 3.2.5. *To give a stability condition on a bounded derived category $\mathcal{D}$ is equivalent to giving a bounded t-structure on $\mathcal{D}$ and a stability function on its heart with the Harder-Narasimhan property.*

Proof. See [Br01] Proposition 5.3. An alternative proof can also be found in [Ar] Proposition 3.1. $\square$

Example 3.2.6. Let $\mathcal{A} = \mathbf{Coh}(C)$, the category of coherent sheaves on a smooth projective curve, C, and let $\mathcal{D}$ be it's bounded derived category. Consider the standard t-structure on $\mathcal{D}$ and the stability function on $\mathcal{A}$ as outlined in Example 3.1.6. Applying Proposition 3.2.5 then gives a stability condition on the bounded derived category $\mathcal{D}$.

3.3 The space of stability conditions

In this section $\mathcal{A}$ will denote an abelian category and $\mathcal{D} = \mathcal{D}^b(\mathcal{A})$ will denote its bounded derived category.

Definition 3.3.1. A slicing $\mathcal{P}$ of the category $\mathcal{D}$ is *locally-finite* if there exists a real number $\eta > 0$ such that for all $t \in \mathbb{R}$ the category $\mathcal{P}((t-\eta, t+\eta)) \subset \mathcal{D}$ is of finite length (See Definition 1.4.17). A stability condition $(Z, \mathcal{P})$ is called locally-finite if the corresponding slicing $\mathcal{P}$ is.

We denote the set of locally-finite slicings of $\mathcal{D}$ by $\mathrm{Slice}(\mathcal{D})$ and the set of locally-finite stability conditions on $\mathcal{D}$ by $\mathrm{Stab}(\mathcal{D})$. It has been shown in [Br01] Section 6 that $\mathrm{Stab}(\mathcal{D})$ can be equipped with a topology induced by the inclusion

$$\mathrm{Slice}(\mathcal{D}) \subset \mathrm{Slice}(\mathcal{D}) \times \mathrm{Hom}_{\mathbb{Z}}(K(\mathcal{D}), \mathbb{C}).$$

The group $\tilde{\mathrm{GL}}^+(2,\mathbb{R})$ plays an important role in the study of $\mathrm{Stab}(\mathcal{D})$. It is the universal cover of the connected group

$$\mathrm{GL}^+(2,\mathbb{R}) = \{A \in \mathrm{GL}(2,\mathbb{R}) |\det A > 0\}.$$

It can be described as follows:

$$\tilde{\mathrm{GL}}^+(2,\mathbb{R}) := \{(A, f) | A \in \mathrm{GL}^+(2,\mathbb{R}) \text{ and } f : \mathbb{R} \to \mathbb{R}$$
$$\text{a compatible increasing map}\}.$$

A map $f : \mathbb{R} \to \mathbb{R}$ is called *compatible* with $A \in \mathrm{GL}^+(2,\mathbb{R})$ if:

(a) $f(\phi+1) = f(\phi)+1$ for all $\phi \in \mathbb{R}$.

(b) For all $\phi \in \mathbb{R}$ and for all $v \in \mathbb{R}^2 \backslash \{0\}$ such that $\exp(i\pi\phi) = \frac{v}{||v||}$, we also have $\exp(i\pi f(\phi)) = \frac{Av}{||Av||}$ (Note that here we identify $\mathbb{C}$ with $\mathbb{R}^2$).

The group structure on $\tilde{\mathrm{GL}}^+(2,\mathbb{R})$ is as follows: for all $(A,f),(B,g) \in \tilde{\mathrm{GL}}^+(2,\mathbb{R})$, $(A,f)\circ(B,g) = (AB, f\circ g)$ and $(f\circ g)(\phi) = f(g(\phi))$. The neutral element is $(I_2, \mathrm{id}_{\mathbb{R}})$, where I_2 denotes the identity 2×2 matrix.

Recall that a *right action* of a group G on a set M is given if for each pair $(m,g) \in M \times G$ there is given an element $m \cdot g \in M$ such that the following two conditions are satisfied:

(a) For all $m \in M$, $m \cdot 1 = m$ where $1 \in G$ is the neutral element

(b) For all $m \in M$ and $g, h \in G, m \cdot (gh) = (m \cdot g) \cdot h$.

Now we have the following important lemma.

Lemma 3.3.2. *The space* $\mathrm{Stab}(\mathcal{D})$ *carries a right action of the group* $\tilde{\mathrm{GL}}^+(2,\mathbb{R})$*.*

Proof. Given a stability condition $\sigma = (Z,\mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$, and a pair $(A,f) \in \tilde{\mathrm{GL}}^+(2,\mathbb{R})$, define a new stability condition $\sigma' = (Z',\mathcal{P}')$ by setting $Z' := A^{-1}\circ Z$ and $\mathcal{P}'(\phi) := \mathcal{P}(f(\phi))$. Let us check that this really is a group action: (a) Clearly $(Z,\mathcal{P})\cdot(I_2,\mathrm{id}_{\mathbb{R}}) = (Z,\mathcal{P})$ for all $(Z,\mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$.

(b) For all $(A,f),(B,g) \in \tilde{\mathrm{GL}}^+(2,\mathbb{R})$ and for all $(Z,\mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$,

$$(Z,\mathcal{P})\cdot((A,f)(B,g)) = (Z,\mathcal{P})\cdot(AB, f\circ g)$$

$$= (B^{-1}\circ A^{-1}\circ Z, \mathcal{P}(f\circ g))$$

and we have

$$((Z,\mathcal{P})\cdot(A,f)))(B,g)) = (A^{-1}\circ Z, \mathcal{P}(f))(B,g)$$

$$= (B^{-1}\circ A^{-1}\circ Z, \mathcal{P}(f\circ g))$$

And so we see that $\mathrm{Stab}(\mathcal{D})$ carries a right action of the group $\tilde{\mathrm{GL}}^+(2,\mathbb{R})$. Note that the semi-stable objects of the stability conditions σ and σ' are the same, but the phases have been relabelled. $\square$

3.4 Examples of spaces of stability conditions

In this section, we give an overview of the known spaces of stability conditions to date that can be given explicitly. We denote the space of locally-finite numerical stability conditions by $\mathrm{Stab}_{\mathcal{N}}(C)$ on $\mathcal{D}(C)$, where $\mathcal{D}(C)$ is the bounded derived category of coherent sheaves on a curve C.

Example 3.4.1. ([Br01] Theorem 9.1) Let C be a smooth projective curve of genus one, then $\tilde{\mathrm{GL}}^+(2,\mathbb{R})$ acts transitively on $\mathrm{Stab}_{\mathcal{N}}(C)$ and

$$\mathrm{Stab}_{\mathcal{N}}(C) \cong \tilde{\mathrm{GL}}^+(2,\mathbb{R}).$$

Example 3.4.2. Burban and Kreussler ([BK]) showed that if C is an irreducible singular curve of genus one, the resulting space of stability conditions is the same as in the smooth case outlined above.

Example 3.4.3. S. Okada ([O]) proved that

$$\mathrm{Stab}_{\mathcal{N}}(\mathbb{P}^1) \cong \mathbb{C}^2.$$

Example 3.4.4. E. Macri ([Ma]) proved that for any curve C of genus $g \geq 2$ one has

$$\mathrm{Stab}_{\mathcal{N}}(C) \cong \tilde{\mathrm{GL}}^+(2,\mathbb{R}).$$

Chapter 4

Coherent systems

We have already looked at stability of vector bundles over a curve. We could also examine stability conditions on higher dimensional manifolds. T. Bridgeland has studied the space of stability conditions on K3 surfaces and on abelian surfaces in [Br03]. However, for the purpose of this thesis I chose to vary the object of which I study stability conditions as opposed to the manifold. We will now examine the stability of coherent systems on a curve. Our aim then is to find out whether or not this notion of stability fits into Bridgeland's framework as outlined in Chapter 3. To begin with, we will outline some preliminaries of coherent systems. Throughout this section, C will denote a smooth projective curve.

4.1 Preliminaries

Definition 4.1.1. A *coherent system of type* (r, d, k) on a smooth projective curve, C, is a pair (E, V) consisting of a coherent sheaf, E, of rank r and degree d over C and a vector subspace $V \subset H^0(E)$ of dimension k.

Definition 4.1.2. A *morphism* of coherent systems $f : (E', V') \to (E, V)$ is a morphism of coherent sheaves $f : E' \to E$ such that $H^0(f)(V') \subset V$.

Let us denote by **CohSys**(C), the category in which the objects are coherent systems on C and the morphisms are morphisms of coherent systems. This category has a zero object, $(0,0)$, which consists of the zero coherent sheaf and the zero vector space. Let us show that $(0,0)$ is the zero object as in Definition 1.4.5. We must show that for any coherent system (E,V) there is precisely one morphism to and from $(0,0)$. Let $f:(0,0)\to(E,V)$ be a morphism in $\mathbf{CohSys}(C)$. By definition, $f:0\to E$ is a morphism of coherent sheaves and $H^0(f)(0)\subset V$. Since 0 is the zero object in $\mathbf{Coh}(C)$, this morphism is unique. Similarly we can show that the morphism $g:(E,V)\to(0,0)$ is unique. Hence $(0,0)$ is the zero object in $\mathbf{CohSys}(C)$.

Definition 4.1.3. A *coherent subsystem* of (E,V) is a coherent system (E',V') such that E' is a subsheaf of E and $V'\subset V\cap H^0(E')$.

Since $\mathbf{Coh}(C)$ is an abelian category, we know the Hom-sets are equipped with the structure of an abelian group such that composition distributes over addition. Thus, the Hom-sets in $\mathbf{CohSys}(C)$ also have this property.

Now, for every pair of objects (E,V) and (E',V') in $\mathbf{CohSys}(C)$, we construct a product as follows:

$$(E,V)\times(E',V'):=(E\oplus E',V\oplus V').$$

Let us show that this really is a product in the sense of Definition 1.4.12. Since $E\oplus E'$ is the product of E and E' in the category of $\mathbf{Coh}(C)$ we know that there exists morphisms

$$E \xleftarrow{\mathrm{pr}_1} E\oplus E' \xrightarrow{\mathrm{pr}_2} E'$$

Clearly $V\oplus V'\subset H^0(E\oplus E')$, hence $\mathrm{pr}_1:E\oplus E'\to E$ and $\mathrm{pr}_2:E\oplus E'\to E'$ are morphisms of coherent systems . Now assume we are given two morphisms of coherent systems

$$(E,V) \xleftarrow{a} (F,W) \xrightarrow{a'} (E',V')$$

where $a:F\to E$ and $a':F\to E'$ are morphisms of coherent sheaves. In $\mathbf{Coh}(C)$, there is a unique morphism $a\oplus a':F\to E\oplus E'$. It remains to show that $H^0(a\oplus a')(W) \subset V\oplus V'$. We know $H^0(a)(W) \subset V$ and $H^0(a')(W) \subset V'$ by assumption, hence

$$H^0(a \oplus a')(W) = \{(a \oplus a')(w) = (a(w), a'(w)) \in H^0(E) \oplus H^0(E')|w \in W\}$$

$$\subset V \oplus V'.$$

Hence $a \oplus a' : (F, W) \to (E \oplus E', V \oplus V')$ is a unique morphism of coherent systems and so we really construcuted a product in $\mathbf{CohSys}(C)$. Thus $\mathbf{CohSys}(C)$ is an additive category (see Definition 1.4.30).

The category $\mathbf{CohSys}(C)$ also has kernels and cokernels as follows: The kernel of the morphism $f : (E', V') \to (E, V)$ is the coherent subsystem $(\ker f, V' \cap H^0(\ker f))$, where $\ker f$ is the usual kernel of the sheaf morphism $f : E' \to E$.

The cokernel of the morphism $f : (E', V') \to (E, V)$ is the coherent system $(\operatorname{coker} f, V'')$ where V'' is the image of V in $H^0(\operatorname{coker} f)$ and $\operatorname{coker} f$ is the sheaf associated to the presheaf $\operatorname{coker}' f$. We must now check that these actually define a kernel and cokernel in $\mathbf{CohSys}(C)$ as outlined in Definitions 1.4.7 and 1.4.8, respectively.

Kernel : Given a morphism $f : (E', V') \to (E, V)$ of coherent systems, we want to show that $i : (\mathcal{K}, W) \to (E', V')$ is the kernel of f, where $(\mathcal{K}, W) := (\ker f, V' \cap H^0(\ker f))$. Clearly from the definitions $(\mathcal{K}, W)$ is itself a coherent system (i.e. $W \subset H^0(\mathcal{K})), i : (\mathcal{K}, W) \to (E', V')$ is a morphism of coherent systems (i.e. $i : \mathcal{K} \to E'$ is a morphism of sheaves and $H^0(i)(W) \subset V'$) and $fi = 0$.

It remains to show that $i : (\mathcal{K}, W) \to (E', V')$ satisfies the universal property, i.e. given a morphism $e : (F, U) \to (E', V')$ such that $fe = 0$, we must show that there exists a unique morphism $e' : (F, U) \to (\mathcal{K}, W)$ such that $ie' = e$

$$\begin{tikzcd}
(\mathcal{K}, W) \arrow[r, "i"] & (E', V') \arrow[r, "f"] & (E, V) . \\
(F, U) \arrow[u, dashed, "e'"] \arrow[ur, "e"'] & &
\end{tikzcd}$$

Consider the morphisms in $\mathbf{Coh}(C)$. Since $\mathcal{K} = \ker f$ in $\mathbf{Coh}(C)$, we know $i : \mathcal{K} \to E'$ satisfies the universal property, i.e. for every $e : F \to E'$ satisfying $fe = 0$, we have the following commutative diagram of coherent sheaves

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{K} \arrow[r, "i"] & E' \arrow[r, "f"] & E \\ & F \arrow[u, "e'"] \arrow[ur, "e"'] & & \end{tikzcd}$$

for a unique e'. This induces linear maps (of vector spaces)

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{K}) \arrow[r, "H^0(i)"] & H^0(E') \arrow[r, "H^0(f)"] & H^0(E) \\ & H^0(F) \arrow[u, "H^0(e')"] \arrow[ur, "H^0(e)"'] & & \end{tikzcd}$$

with $U \subset H^0(F), W \subset H^0(\mathcal{K}), V' \subset H^0(E'), V \subset H^0(E)$. The above diagram is commutative and the first row is exact. We know that $H^0(e)(U) \subset V'$, since e is a morphism of coherent systems. So all that remains to show now is that $H^0(e')(U) \subset W$, i.e. that e' really is a morphism of coherent systems.

For any $u \in U \subset H^0(F)$ we have $H^0(f)H^0(e)(u) = 0$ (because $f \circ e = 0$, hence $H^0(f) \circ H^0(e) = H^0(f \circ e) = 0$), i.e. $H^0(e)(u) \in \ker(H^0(f))$. Hence $H^0(e)(u) \in V' \cap \ker(H^0(f))$. But because of the exactness of the first row, $\ker(H^0(f)) = H^0(\mathcal{K})$ (considered as a subspace of $H^0(E')$). This gives $H^0(e')(u) = H^0(i)H^0(e')(u) = H^0(e)(u) = H^0(e)(u) \in V' \cap H^0(\mathcal{K}) = W$ as required.

Cokernel : Now given a morphism $f : (E', V') \to (E, V)$ we want show that $p : (E, V) \to (\mathcal{C}, W)$ is the cokernel of f, where $(\mathcal{C}, W) := (\operatorname{coker} f, H^0(p)(V))$. Clearly from the definitions $(\mathcal{C}, W)$ is itself a coherent system (i.e. $W \subset H^0(\mathcal{C})), p : (E, V) \to (\mathcal{C}, W)$ is a morphism of coherent systems (i.e. $p : E \to \mathcal{C}$ is a morphism of coherent sheaves and $H^0(p)(V) \subset W$) and $pf = 0$.

It remains to show that $p : (E, V) \to (\mathcal{C}, W)$ satisfies the universal property, i.e. given a morphism $g : (E, V) \to (F, U)$ such that $gf = 0$, we must show that there exists a unique morphism $g' : (\mathcal{C}, W) \to (F, U)$ such that $g = g'p$

$$\begin{tikzcd}
(E', V') \arrow[r, "f"] & (E, V) \arrow[r, "p"] \arrow[d, "g"'] & (\mathcal{C}, W) . \arrow[dl, dashed, "g'"] \\
& (F, U) &
\end{tikzcd}$$

Consider the morphisms in $\mathbf{Coh}(C)$. Since $\mathcal{C} = \operatorname{coker} f$ in $\mathbf{Coh}(C)$, we know $p : E \to \mathcal{C}$ satisfies the universal property, i.e. for every $g : E \to F$ satisfying $gf = 0$, we have the following commutative diagram

$$\begin{tikzcd}
E' \arrow[r, "f"] & E \arrow[r, "p"] \arrow[d, "g"'] & \mathcal{C} \arrow[dl, "g'"] \\
& F &
\end{tikzcd}$$

for a unique g'. All that remains to show now is that $H^0(g')(W) \subset U$, i.e. that g' really is a morphism of coherent systems. Consider first the induced maps on the vector spaces as follows:

$$\begin{tikzcd}
H^0(E') \arrow[r, "H^0(f)"] & H^0(E) \arrow[r, "H^0(p)"] \arrow[d, "H^0(g)"'] & H^0(\mathcal{C}) \arrow[dl, "H^0(g')"] \\
& H^0(F) &
\end{tikzcd} \tag{4.1}$$

with $V' \subset H^0(E'), V \subset H^0(E), W \subset H^0(\mathcal{C})$ and $U \subset H^0(F)$. Since p and g are morphisms of coherent systems, we know that $H^0(p)(V) \subset W$ and $H^0(g)(V) \subset U$. From the commutativity of (4.1), if $v \in V$ we then get $H^0(g')H^0(p)(v) = H^0(g)(v)$. Now for any $w \in W \subset H^0(\mathcal{C})$, we know by construction of W that there exists $v \in V$ such that $H^0(p)(v) = w$. Hence, we have $H^0(g')(w) = H^0(g)(v) \in U$ as required.

4.2 Stability

In contrast to stability of vector bundles on curves, the stability notion for coherent systems depends on a real parameter α. This notion of stability permits the construction of moduli spaces.

Definition 4.2.1. For any real number α, the α-*slope* of a coherent system (E, V) of type (r, d, k) is defined by

$$\mu_\alpha(E, V) := \frac{d}{r} + \alpha\frac{k}{r}.$$

A coherent system (E, V) is called α-*stable* (resp. α-semi-stable) if

$$\mu_\alpha(E', V') < \mu_\alpha(E, V), \quad \text{(resp. } \mu_\alpha(E', V') \leq \mu_\alpha(E, V))$$

for every nontrivial coherent subsystem (E', V') of (E, V), i.e. every coherent subsystem other than $(0, 0)$ and (E, V) itself.

4.2.1 Moduli spaces of (semi-)stable coherent systems

Here we outline some of what is available in the literature about the moduli spaces of (semi-)stable coherent systems. Note that all coherent systems (E, V) considered in this and the following subsection consist of a vector bundle (i.e. a locally free sheaf), E and a vector subspace $V \subset H^0(E)$.

The α-stable coherent systems of type (r, d, k) on C form a moduli space, which we will denote $G(\alpha; r, d, k)$. The α-range for which α-stable coherent systems exist is divided into a finite number of open intervals such that the moduli spaces $G(\alpha; r, d, k)$ for any two values of α inside the same interval coincide. In the case of C an elliptic curve, Lange and Newstead ([LN]) have given a complete description of these moduli spaces.

We denote the moduli space of stable vector bundles of rank r and degree d on an elliptic curve C by $M(r, d)$. Now, if $k \geq 1$ and $\alpha \leq 0$, α-stable coherent systems of type (r, d, k) do not exist (See [BGMN], Section 2.1). The main results of [LN] can be summarised by the following theorem:

Theorem 4.2.2. *Let C be an elliptic curve and suppose $r \geq 1, k \geq 0$. Then:*

(a) $G(\alpha; r, d, 0) \cong M(r, d)$ for all α. In particular $G(\alpha; r, d, 0)$ is non-empty if and only if $\gcd(r, d) = 1$*;*

(b) For $\alpha > 0$ and $k \geq 1, G(\alpha; 1, d, k)$ is independent of α and is non-empty if and only if either $d = 0, k = 1$ or $k \leq d$;

(c) For $\alpha > 0, r \geq 2$ and $k \geq 1, G(\alpha; r, d, k) \neq \emptyset$ if and only if $(r-k)\alpha < d$ and either $k < d$ or $k = d$ and $\gcd(r, d) = 1$.

4.2.2 Bounds for α

We know that if $k \geq 1, \alpha$-stable coherent systems on an elliptic curve, C, exist only for $\alpha > 0$. The following theorem ([LN], Theorem 4.4) tells us information about the bounds for α, when examining moduli spaces of α-stable coherent systems.

Theorem 4.2.3. *The set*

$$I(r, d, k) := \{\alpha | G(\alpha; r, d, k) \neq \emptyset\}$$

is an open interval (possibly infinite or empty). Moreover, if $I(r, d, k) \neq \emptyset$, there exists a coherent system of type (r, d, k) which is α-stable for all $\alpha \in I(r, d, k)$.

To obtain the range of α for which $G(\alpha; r, d, k) \neq \emptyset$, by Theorem 4.2.3, it is enough to find the upper and lower bound for α. These bounds are given by the following two theorems.

Theorem 4.2.4. *([LN] Theorem 5.1) Suppose $k > 0$ and either $k < d$ or $k = d$ and $\gcd(r, d) = 1$. Then*

$$\inf\{\alpha | G(\alpha; r, d, k) \neq \emptyset\} = 0.$$

Theorem 4.2.5. *([LN] Theorem 5.2) Suppose $0 < k < r$ and that either $k < d$ or $k = d$ and $\gcd(r, d) = 1$. Then*

$$\sup\{\alpha | G(\alpha; r, d, k\} \neq \emptyset\} = \frac{d}{r-k}.$$

We call $\alpha > 0$ a *critical value* if there exists a proper coherent subsystem (E', V') of (E, V) such that $\frac{k'}{r'} \neq \frac{k}{r}$ but $\mu_\alpha(E', V') = \mu_\alpha(E, V)$. We also call 0 a critical value.

If the critical values of α are denoted α_i, starting with $\alpha_0 = 0$, the α-range is divided into (α_i, α_{i+1}). Within the interval (α_i, α_{i+1}), the property of α-stability is independent of α. This means that for any two values α, α' in the interval $(\alpha_i, \alpha_{i+1}), G(\alpha; r, d, k) = G(\alpha'; r, d, k)$ ([BGMMN] Definition 1.2).

Proposition 4.2.6. *([BGMN] Proposition 4.2) Let $k \geq r$. Then there is a critical value, denoted by α_L such that the α-range is divided into a finite set of intervals bounded by critical values such that*

$$0 < \alpha_0 < \alpha_1 < \cdots < \alpha_L < \infty$$

where, for any two $\alpha, \alpha' \in (\alpha_L, \infty), G(\alpha; r, d, k) = G(\alpha'; r, d, k)$.

So now we have seen that the moduli space of semi-stable coherent systems depends on a parameter. It would be very interesting to see if this fits into the framework of Bridgeland stability conditions and if so, what role the parameter plays in this framework.

4.3 Bridgeland stability

In order to view α-stability as a Bridgeland stability condition, the first step would be to check whether or not the category of coherent systems on a smooth projecitve curve, $\mathbf{CohSys}(C)$, is abelian. We know $\mathbf{CohSys}(C)$ is an additive category in which every morphism has a kernel and a cokernel. If we also knew that all morphisms in $\mathbf{CohSys}(C)$ are strict, then by Proposition 1.4.34, we would get that $\mathbf{CohSys}(C)$ is an abelian category. We have seen the definition of a strict morphism (Definition 1.4.10) in Section 1.4. Let us now give a more workable description of a strict morphism in $\mathbf{CohSys}(C)$.

Lemma 4.3.1. *A morphism $f : (E', V') \to (E, V)$ of coherent systems is a strict morphism if and only if $H^0(f)(V') = V \cap H^0(\operatorname{im} f)$.*

Proof. Because the category, $\mathbf{Coh}(C)$ is an abelian category, $f : E' \to E$ is strict. This means that $f : E' \to \operatorname{im}(f)$ is the cokernel of $\ker(f) \subset E'$ and $\operatorname{im}(f) \subset E$ is the kernel of $E \to \operatorname{coker}(f)$. Using our constructions of ker and coker in the category of coherent systems, we get that the kernel of $f : (E', V') \to (E, V)$ is $(\ker(f), V' \cap H^0(\ker f))$ and that its cokernel is $(\operatorname{coker}(f), H^0(g)(V))$. Now the cokernel of $(\ker(f), V' \cap H^0(\ker f)) \subset (E', V')$ is $(\operatorname{im} f, H^0(f)(V'))$ and the kernel of $(E, V) \to (\operatorname{coker}(f), H^0(g)(V))$ is $(\operatorname{im}(f), H^0(\operatorname{im}(f)) \cap V) \subset (E, V)$. So we see that $f : (E', V') \to (E, V)$ is a strict morphism if and only if $H^0(\operatorname{im}(f)) \cap V = H^0(f)(V')$. □

However, not all morphisms in $\mathbf{CohSys}(C)$ are strict. Let us look at an example of a nonstrict morphism.

Example 4.3.2. Consider two coherent systems, $(\mathcal{O}^{\oplus 2}, 0)$ and $(\mathcal{O}, H^0(\mathcal{O}))$. Let $\mathrm{pr}_1 : \mathcal{O}^{\oplus 2} \to \mathcal{O}$ be a morphism of coherent sheaves given by projection to the first factor. This is an epimorphism in $\mathbf{Coh}(C)$, i.e. $\operatorname{im}(p_1) = \mathcal{O}$. So we have a morphism of coherent systems

$$(\mathcal{O}^{\oplus 2}, 0) \xrightarrow{\mathrm{pr}_1} (\mathcal{O}, H^0(\mathcal{O})).$$

We want to know if this morphism is strict. Now $H^0(\mathrm{pr}_1)(0) = 0$, since $H^0(\mathrm{pr}_1)$ is a linear map. Then we have $H^0(\mathcal{O}) \cap H^0(\operatorname{im}(\mathrm{pr}_1)) = H^0(\mathcal{O})$. So clearly $0 \neq H^0(\mathcal{O})$, hence pr_1 is not a strict morphism of coherent systems.

Thus we conclude that $\mathbf{CohSys}(C)$ is not an abelian category. So now we define a new category, denoted $\mathbf{CohSys^{st}}(C)$ in which the objects are again coherent systems but the morphisms are all the strict morphisms. It seems plausible that this category is abelian (See [P02], Chapter 4, Section 4.1). Let us check the details. We must first check if $\mathbf{CohSys^{st}}(C)$ is an additive category. It turns out that $\operatorname{Hom}_{\mathbf{CohSys^{st}}(C)}((E', V'), (E, V))$ is not always an additive subgroup of $\operatorname{Hom}_{\mathbf{CohSys}(C)}((E', V'), (E, V))$. We illustrate this by the following example.

Example 4.3.3. Consider the coherent system $(\mathcal{O},0)$ and let (E,V) be another coherent system. Then

$$\mathrm{Hom}_{\mathbf{CohSys^{st}}(C)}((\mathcal{O},0),(E,V)) = \{f : \mathcal{O} \to E | H^0(f)(0) = V \cap H^0(\mathrm{im}\, f)\}$$

We know from Proposition 2.1.18 that $\mathrm{Hom}_{\mathbf{Coh}(C)}(\mathcal{O},E) \cong H^0(E)$. Under this identification $f \in \mathrm{Hom}_{\mathbf{Coh}(C)}(\mathcal{O},E)$ corresponds to $H^0(f)(1) = s \in H^0(E)$. Now $H^0(\mathrm{im}\, f) = \mathbb{C}\cdot s$, hence $V \cap H^0(\mathrm{im}\, f) = 0$ if and only if $s \notin V$ or $s = 0$. So we have

$$\mathrm{Hom}_{\mathbf{CohSys^{st}}(C)}((\mathcal{O},0),(E,V)) = \{s \in H^0(E) | s = 0 \text{ or } s \notin V\}.$$

$$= (H^0(E)\backslash V) \cup \{0\}$$

If $w \in H^0(E)\backslash V$ and $v \in V$, then $w-v \in H^0(E)\backslash V$. But $w-(w-v) = v \in V$. If $0 \neq V$ and $V \neq H^0(E)$, this shows that $(H^0(E)\backslash V)\backslash\{0\}$ is not an additive subgroup of $H^0(E)$. For example, if $E = \mathcal{O}^{\oplus 2}$ and $V \subset H^0(E) \cong \mathbb{C}^2$ any one dimensional subspace, then

$$\mathrm{Hom}_{\mathbf{CohSys^{st}}(C)}((\mathcal{O},0),(\mathcal{O}^{\oplus 2},V)) \subset \mathrm{Hom}_{\mathbf{CohSys}(C)}(\mathcal{O},0),(\mathcal{O}^{\oplus 2},V))$$

is not an additive subgroup.

Hence $\mathbf{CohSys^{st}}(C)$ is not an additive subcategory of $\mathbf{CohSys}(C)$. However, we cannot conclude that $\mathbf{CohSys^{st}}(C)$ is not an additive category, as it may have a different additive structure. In order to show that $\mathbf{CohSys^{st}}(C)$ is not an abelian category, we go back to the axioms given in Definition 1.4.27. We first show the existence of the zero object and of kernels and cokernels but then we show that products do not exist in general in $\mathbf{CohSys^{st}}(C)$.

1. The category $\mathbf{CohSys^{st}}(C)$ *has a zero object, namely* $(0,0)$.

3. We must now show that every morphism has a kernel and a cokernel.

Kernel : We have seen in Section 4.1 that every morphism in $\mathbf{CohSys}(C)$ has a kernel. If $f : (E',V') \to (E,V)$ is a strict morphism, its kernel in $\mathbf{CohSys}(C)$ is $(\mathcal{K},W) = (\ker f, V' \cap H^0(\ker f))$. Now we want to show that $i : (\mathcal{K},W) \to (E',V')$ is also the kernel of f in $\mathbf{CohSys^{st}}(C)$. Firstly, it is clear that i is a strict morphism, i.e. $H^0(i)(W) = V' \cap H^0(\mathrm{im}(i))$, since by definition $W := V' \cap H^0(\mathcal{K})$ and $i : \mathcal{K} \subset E'$ is just inclusion.

It remains to show that $i : (\mathcal{K}, W) \to (E', V')$ satisfies the universal property, i.e. given strict a morphism $e : (F, U) \to (E', V')$ such that $fe = 0$, we must show that there exists a unique strict morphism $e' : (F, U) \to (\mathcal{K}, W)$ such that $ie' = e$

$$\begin{tikzcd}
(\mathcal{K}, W) \arrow[r, "i"] & (E', V') \arrow[r, "f"] & (E, V) . \\
(F, U) \arrow[u, dashed, "e'"] \arrow[ur, "e"'] & &
\end{tikzcd}$$

We know that there is a unique morphism e' in $\mathbf{CohSys}(C)$ with the property outlined above so we must just show that e' is a strict morphism, i.e. $H^0(e')(U) = W \cap H^0(\mathrm{im}(e'))$. Consider the following commutative diagram in $\mathbf{Coh}(C)$

$$\begin{tikzcd}
\ker(f) \arrow[r, hook] & E' \\
F \arrow[u, "e'"] \arrow[ur, "e"'] &
\end{tikzcd} \tag{4.2}$$

and the induced morphisms on the vector spaces

$$\begin{tikzcd}
W = H^0(\ker(f)) \cap V' \arrow[r, hook] & V' \\
U \arrow[u, "H^0(e')"] \arrow[ur, "H^0(e)"'] &
\end{tikzcd}$$

From this we know that $H^0(e)(U) = H^0(e')(U)$ and from (4.2) that $\mathrm{im}\, e = \mathrm{im}\, e' \subset \ker(f)$. This implies that $H^0(\mathrm{im}(e)) = H^0(\mathrm{im}(e')) \subset H^0(\ker f)$. This in turn implies that $W \cap H^0(\mathrm{im}(e')) = V' \cap H^0(\mathrm{im}(e'))$, but $\mathrm{im}(e') = \mathrm{im}(e)$ so we have $W \cap H^0(\mathrm{im}(e')) = V' \cap H^0(\mathrm{im}(e))$. Since e is strict we know $H^0(e)(U) = V' \cap H^0(\mathrm{im}(e))$, so this gives us $H^0(e')(U) = H^0(e)(U) = W \cap H^0(\mathrm{im}(e'))$ and so e' is a strict morphism.

Cokernel : We have seen in Section 4.1 that every morphism has a cokernel in $\mathbf{CohSys}(C)$. If $f : (E', V') \to (E, V)$ is a strict morphism, then its cokernel in $\mathbf{CohSys}(C)$ is given by $p : (E, V) \to (\mathcal{C}, W)$, where $(\mathcal{C}, W) := (\operatorname{coker} f, H^0(p)(V))$. We now need to show that $p(E, V) \to (\mathcal{C}, W)$ is also the cokernel of f in $\mathbf{CohSys}^{\mathbf{st}}(C)$. The proof is similar to the kernel case so we do not repeat the details.

Lemma 4.3.4. *There does not exist a product of $(\mathcal{O}, H)$ and $(\mathcal{O}, H)$ in $\mathbf{CohSys}^{\mathbf{st}}(C)$, where $H := H^0(\mathcal{O}) \cong \mathbb{C}$.*

Proof. Assume $(\mathcal{F}, V)$ is the product of $(\mathcal{O}, H)$ and $(\mathcal{O}, H)$. It comes with two strict morphisms $(\mathcal{O}, H) \xleftarrow{q_2} (\mathcal{F}, V) \xrightarrow{q_1} (\mathcal{O}, H)$ such that the universal property for the product is satisfied (see again Definition 1.4.12).

Let $\mathrm{pr}_i : \mathcal{O}^{\oplus 2} \to \mathcal{O}$ denote the first and second projections for $i = 1, 2$. Then, $\mathrm{pr}_i : (\mathcal{O}^{\oplus 2}, H^0(\mathcal{O}^{\oplus 2})) \to (\mathcal{O}, H)$ are strict for $i = 1, 2$. Hence the universal property of $(\mathcal{F}, V)$ implies the existence of a (unique) strict morphism $a : (\mathcal{O}^{\oplus 2}, H^0(\mathcal{O}^{\oplus 2})) \to (\mathcal{F}, V)$ such that $q_i \circ a = p_i (i = 1, 2)$. This is illustrated by the following diagram

$$\begin{tikzcd}
 & (\mathcal{O}, H) \\
(\mathcal{O}^{\oplus 2}, H^0(\mathcal{O}^{\oplus 2})) \arrow[ur, "\mathrm{pr}_1"] \arrow[r, "a"] \arrow[dr, "\mathrm{pr}_2"'] & (\mathcal{F}, V) \arrow[u, "q_1"'] \arrow[d, "q_2"] \\
 & (\mathcal{O}, H)
\end{tikzcd}$$

Let $\mathcal{K} := \ker(a)$ in $\mathbf{Coh}(C)$. Then $\mathrm{pr}_i |_{\mathcal{K}} = q_i \circ a_i|_{\mathcal{K}} = 0$. The universal property for the product of $\mathcal{O}^{\oplus 2}$ in $\mathbf{Coh}(C)$ implies $\mathcal{K} = 0$. Hence by Lemma 1.4.32, a is a monomorphism in $\mathbf{Coh}(C)$. It is easy to see that a strict morphism which is a monomorphism in $\mathbf{Coh}(C)$ is also a monomorphism in $\mathbf{CohSys}^{\mathbf{st}}(C)$ and hence a is a monomorphism in $\mathbf{CohSys}^{\mathbf{st}}(C)$.

On the other hand, the universal property of $\mathcal{O}^{\oplus 2}$ in $\mathbf{Coh}(C)$ implies the existence of a morphism $\mathcal{F} \xrightarrow{b} \mathcal{O}^{\oplus 2}$ in $\mathbf{Coh}(C)$ which satisfies $\mathrm{pr}_i \circ b = q_i (i = 1, 2)$, so $b \circ a = \mathrm{id}_{\mathcal{O}^{\oplus 2}}$ (again by universality of the product $\mathcal{O}^{\oplus 2}$ in $\mathbf{Coh}(C)$). This shows (as in Definition 2.1.22) that the sequence

$$0 \longrightarrow \mathcal{O}^{\oplus 2} \xrightarrow{a} \mathcal{F} \longrightarrow \operatorname{coker} a \longrightarrow 0$$

splits, i.e. $\mathcal{F} \cong \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{G}$, where $\mathcal{G} := \operatorname{coker} a$. This gives us a monomorphism in $\mathbf{Coh}(C)$, $\mathcal{G} \xhookrightarrow{\gamma} \mathcal{F}$ and with $W := V \cap H^0(\mathcal{G}), \gamma : (\mathcal{G}, W) \to (\mathcal{F}, V)$ is a monomorphism in $\mathbf{CohSys^{st}}(C)$

Now we want to show that $\mathcal{G} = 0$. First assume that $W = 0$. The morphisms $\gamma_i := q_i \circ \gamma : (\mathcal{G}, 0) \to (\mathcal{O}, H)$ are strict. Hence $H^0(\operatorname{im}(\gamma_i)) = 0$. If $\tilde{\gamma} : \mathcal{G} \to \mathcal{O}^{\oplus 2}$ is the morphism which is defined by the universality of product in $\mathbf{Coh}(C)$, we also obtain $H^0(\operatorname{im}(\tilde{\gamma})) = 0$. In particular, $\tilde{\gamma} : (\mathcal{G}, 0) \to (\mathcal{O}^{\oplus 2}, H^0(\mathcal{O}^{\oplus 2}))$ is strict, hence $a \circ \tilde{\gamma} : (\mathcal{G}, 0) \to (\mathcal{F}, V)$ is strict and satisfies $q_i \circ a \circ \tilde{\gamma} = \gamma_i$. This contradicts uniqueness in the definition of the product $(\mathcal{F}, V)$ because $a \circ \tilde{\gamma} \neq \gamma$.

Now let us assume that $W \neq 0$. Let $w_1, \ldots, w_m$ be a basis of W. Then, the sheaf of sections, $\mathcal{W}$, of the trivial vector bundle, $C \times W$, is isomorphic to $\mathcal{O}_C^{\oplus m}$. We obtain a homomorphism $\beta : \mathcal{W} \cong \mathcal{O}_C^{\oplus m} \to \mathcal{G}$ of $\mathcal{O}_C$-modules by

$$\beta_U(f_1, \ldots, f_m) = \sum_{i=1}^{m} f_i \cdot w_i|_U$$

for any open set $U \subset C$. The homomorphism $H^0(\beta) : H^0(\mathcal{W}) \to H^0(\mathcal{G})$ identifies $H^0(\mathcal{W})$ with $W \subset H^0(\mathcal{G})$.

The composition $\beta_i := q_i \circ \gamma \circ \beta : \mathcal{W} \to \mathcal{O}$ induces a morphism in $\mathbf{Coh}(C), \eta : \mathcal{W} \to \mathcal{O}^{\oplus 2}$ with $\mathrm{pr}_i \circ \eta = \beta_i$. So η is given by a constant matrix and this implies $H^0(\eta)(H^0(\mathcal{W})) = H^0(\operatorname{im} \eta) \subset H^0(\mathcal{O}^{\oplus 2})$. Hence $\eta : (\mathcal{W}, H^0(\mathcal{W})) \to (\mathcal{O}^{\oplus 2}, H^0(\mathcal{O}^{\oplus 2}))$ is a strict morphism. Again $a \circ \eta : (\mathcal{W}, W) \to (\mathcal{F}, V)$ satisfies $q_i \circ a \circ \eta = \beta_i$, which is also true for $\gamma \circ \beta : (\mathcal{W}, W) \to (\mathcal{F}, V)$ but $a \circ \eta \neq \gamma \circ \beta$ is a contradiction to the uniqueness of the universality of $(\mathcal{F}, V)$. This implies that $\mathcal{G} = 0$.

Hence $\mathcal{F} = \mathcal{O}^{\oplus 2}$ and $V = H^0(\mathcal{O}^{\oplus 2})$. But now we obtain a contradiction with the universal property of $(\mathcal{F}, V)$ as follows. Consider a coherent system $(\mathcal{O}^{\oplus 2}, L)$ where $L \subset H^0(\mathcal{O}^{\oplus 2})$ is a one-dimensional linear subspace such that $L \neq \ker(H^0(\mathrm{pr}_1))$ and $L \neq \ker(H^0(\mathrm{pr}_2))$. These assumptions ensure that L maps surjectively onto $H = H^0(\mathcal{O})$ under $H^0(q_1)$ and $H^0(q_2)$. Consider the following diagram

$$\begin{tikzcd} & & (\mathcal{O}, H) \\ (\mathcal{O}^{\oplus 2}, L) \arrow[urr, "\mathrm{pr}_1"] \arrow[drr, "\mathrm{pr}_2"'] \arrow[r, "a"] & (\mathcal{O}^{\oplus 2}, H^0(\mathcal{O}^{\oplus 2})) \arrow[ur, "q_1"'] \arrow[dr, "q_2"] & \\ & & (\mathcal{O}, H) \end{tikzcd}$$

where a is the unique strict morphism given by the universal property. By the universal property of $\mathcal{O}^{\oplus 2}$ in $\mathbf{Coh}(C)$ we get that a must be the identity. But then we would have that $H^0(a)(L) = H^0(\mathcal{O}^{\oplus 2})$ which is impossible, since L is one-dimensional and $H^0(\mathcal{O}^{\oplus 2})$ is two-dimensional. This implies that a product $(\mathcal{O}, H) \times (\mathcal{O}, H)$ does not exist in $\mathbf{CohSys^{st}}(C)$. $\square$

Hence we conclude that there does not exist a product for every pair of objects in the category $\mathbf{CohSys^{st}}(C)$ and so it fails at Axiom 2 for a category to be abelian (Definition 1.4.27).

To sum up, we have shown that $\mathbf{CohSys}(C)$ is not an abelian category, as not all morphisms are strict in this category. We then showed that $\mathbf{CohSys^{st}}(C)$ is also not an abelian category because there does not exist a product for every pair of objects. One possible way of overcoming this problem is to consider $\mathbf{CohSys}(C)$ as an additive subcategory of an appropriate abelian category, $\mathcal{C}(C)$. In [KN] such a category is defined. Its objects consist of a finite dimensional vector space V, a coherent sheaf, $\mathcal{E}$ and a sheaf morphim $\varphi : \mathcal{V} \to \mathcal{E}$, where $\mathcal{V}$ is the sheaf of sections of the trivial vector bundle $C \times V$. A morphism in this category is a commutative diagram of morphisms of coherent sheaves

$$\begin{tikzcd} \mathcal{V}_1 \arrow[r, "\varphi_1"] \arrow[d] & \mathcal{E}_1 \arrow[d] \\ \mathcal{V}_2 \arrow[r, "\varphi_2"] & \mathcal{E}_2 \end{tikzcd}$$

where $\mathcal{V}_i$ is the sheaf of sections of the trivial vector bundle $C \times V_i$ for $i = 1, 2$.

This category $\mathcal{C}(C)$ is a full subcategory of the category of holomorphic triples which is studied in the next chapter.

Chapter 5

Holomorphic triples

We will now study stability of another another object, namely a holomorphic triple, as an extenstion of coherent sheaves in the hope of defining Bridgeland stability conditions on triples. We begin with some preliminaries.

5.1 Preliminaries

Let us first give a definition of a triple.

Definition 5.1.1. A *holomorphic triple* $T = (E_1, E_2, \phi)$ on X, a compact Riemann surface, consists of two coherent sheaves, E_1 and E_2, on X and a sheaf morphism $\phi : E_1 \to E_2$.

Definition 5.1.2. A *homomorphism* of triples, $f : T' \to T$ from $T' = (E_1', E_2', \phi')$ to $T = (E_1, E_2, \phi)$ is a commutative diagram

$$\begin{array}{ccc} E_1' & \xrightarrow{\phi'} & E_2' \\ \downarrow & & \downarrow \\ E_1 & \xrightarrow{\phi} & E_2 \end{array}$$

where the vertical arrows are ($\mathcal{O}_X$-linear) sheaf morphisms.

We denote by $\mathcal{T}\mathbf{Coh}(X)$, the category in which the objects are triples and the morphisms are homomorphisms of triples. This category has a zero object, i.e. the zero triple $T = 0$ obtained by taking $E_1 = E_2 = 0$, the zero object in $\mathbf{Coh}(X)$. We check that this really is the zero object of $\mathcal{T}\mathbf{Coh}(X)$ as in Definition 1.4.5. We need that for any triple, there is exactly one morphism to and from $T = 0$. Let $f : 0 \to T'$ be a morphism in $\mathcal{T}\mathbf{Coh}(X)$, i.e. we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ E_1' \arrow[r, "\phi'"] & E_2' \end{tikzcd}$$

in which the vertical arrows are the zero morphisms in $\mathbf{Coh}(X)$. Hence f is unique. Similarly we can show that $g : T' \to 0$ is unique and so we have $T = 0$ is the zero object in $\mathcal{T}\mathbf{Coh}(X)$.

Definition 5.1.3. A triple $T' = (E_1', E_2', \phi')$ is a *subtriple* of $T = (E_1, E_2, \phi)$ if:

(a) E_i' is a coherent subsheaf of E_i, for $i = 1, 2$ and

(b) we have the commutative diagram

$$\begin{tikzcd} E_1' \arrow[r, "\phi'"] \arrow[d, hook] & E_2' \arrow[d, hook] \\ E_1 \arrow[r, "\phi"] & E_2 \end{tikzcd}$$

The category, $\mathcal{T}\mathbf{Coh}(X)$ has kernels and cokernels, described as follows:

Kernel : Let $T = (E_1, E_2, \phi)$ and $T' = (E_1', E_2', \phi')$ be two objects in $\mathcal{T}\mathbf{Coh}(X)$ and let $f : T' \to T$ be a morphism, given by the following commutative diagram

$$\begin{tikzcd} E_1' \arrow[r, "\phi'"] \arrow[d, "f_1"'] & E_2' \arrow[d, "f_2"] \\ E_1 \arrow[r, "\phi"] & E_2 \end{tikzcd}$$

Let the kernel of f be the morphism, $i : K \to T'$, with $K = (K_1, K_2, \psi)$, where $K_1 := \ker(E'_1 \to E_1)$ and $K_2 := \ker(E'_2 \to E_2)$ and $\psi : K_1 \to K_2$ is the unique morphism induced by ϕ', i.e. ϕ' restricted to K_1.

Let us show that $i : K \to T'$ is a morphism of triples, i.e. that we have a commutative diagram

$$\begin{tikzcd} K_1 \arrow[r, "\psi"] \arrow[d, hook, "i_1"'] & K_2 \arrow[d, hook, "i_2"] \\ E'_1 \arrow[r, "\phi'"] & E'_2 \end{tikzcd} \tag{5.1}$$

with i_1 and i_2 being sheaf morphisms. This follows from the universal property of the kernel of $f_2 : E'_2 \to E_2$ as follows: we know that for any morphism $e : K_1 \to E_2$ such that $f_2 \circ e = 0$, we have $i_2 \circ \psi = e$ for a unique $\psi : K_1 \to K_2$. Now if we let $e := \phi' \circ i_1$, then we have $f_2 \circ \phi' \circ i_1 = \phi \circ f_1 \circ i_1 = 0$, since K_1 is the kernel of f_1 (hence $f_1 \circ i_1 = 0$). Hence, we get $i_2 \circ \psi = \phi' \circ i_1$, i.e. $i : K \to T'$ really is a morphism of triples.

Now we need to verify that the morphism, $i : K \to T'$ satisfies the universal property of a kernel, i.e. given another morphism $g : U \to T'$ with $U = (U_1, U_2, \varphi)$

$$\begin{tikzcd} U_1 \arrow[r, "\varphi"] \arrow[d, "g_1"'] & U_2 \arrow[d, "g_2"] \\ E'_1 \arrow[r, "\phi'"] \arrow[d, "f_1"'] & E'_2 \arrow[d, "f_2"] \\ E_1 \arrow[r, "\phi"'] & E_2 \end{tikzcd}$$

such that $fg = 0$, we must show that this factors uniquely through K as $g = ie'$ for a unique $e' : U \to K$, i.e. that there exists a unique e' such that the following diagram commutes

$$\begin{tikzcd}
U_1 \arrow[r, dashed, "e_1'"] \arrow[dr, "g_1"'] & K_1 \arrow[r, "\psi"] \arrow[d, "i_1"] & K_2 \arrow[d, "i_2"] & U_2 \arrow[l, dashed, "e_2'"'] \arrow[dl, "g_2"] \\
& E_1' \arrow[r, "\phi'"] \arrow[d, "f_1"'] & E_2' \arrow[d, "f_2"] & \\
& E_1 \arrow[r, "\phi"] & E_2 &
\end{tikzcd}$$

We know e_1' and e_2' exist and are unique in $\mathbf{Coh}(X)$, as this is an abelian category. So it remains to show that e' is a morphism in $\mathcal{T}\mathbf{Coh}(X)$, i.e. that

$$\begin{tikzcd}
U_1 \arrow[r, "\varphi"] \arrow[d, "e_1'"'] & U_2 \arrow[d, "e_2'"] \\
K_1 \arrow[r, "\psi"] & K_2
\end{tikzcd}$$

commutes. This follows from the universal property of the kernel of $f_2 : E_2' \to E_2$ as follows: we know we have $h : U_1 \to E_2'$ such that $f_2 h = 0$, where $h = \phi' \circ i_1 \circ e_1'$ (we know $f_2 h = 0$ since $f_2 \circ \phi' \circ i_1 \circ e_1' = \phi \circ f_1 \circ i_1 \circ e_1 = 0$, as i_1 is the kernel of f_1). The universal property then tells us that there exists a unique morphism from $U_1 \to K_2$ whose composition with i_2 is equal to h, hence $e_2' \circ \varphi = \psi \circ e_1'$ is this unique morphism.

Cokernel : Let $T = (E_1, E_2, \phi)$ and $T' = (E_1', E_2', \phi')$ be two objects in $\mathcal{T}\mathbf{Coh}(X)$ and $f : T' \to T$ a morphism, given by the following commutative diagram

$$\begin{tikzcd}
E_1' \arrow[r, "\phi'"] \arrow[d, "f_1"'] & E_2' \arrow[d, "f_2"] \\
E_1 \arrow[r, "\phi"] & E_2
\end{tikzcd}$$

Let the cokernel of f be the morphism $p : T \to C$, with $C = (C_1, C_2, \varphi)$, where $C_1 := \mathrm{coker}(E_1' \to E_1)$ and $C_2 := \mathrm{coker}(E_2' \to E_2)$ and $\varphi : C_1 \to C_2$ the morphism induced by ϕ.

We must verify that $p : T \to C$ really is a morphism in $\mathcal{T}\mathbf{Coh}(X)$ and that the morphism, $i : K \to T'$ satisfies the universal property of a cokernel, i.e. given another morphism $g : T \to U$ with $U = (U_1, U_2, \psi)$

$$\begin{tikzcd}
E_1' \arrow[r, "\phi'"] \arrow[d, "f_1"'] & E_2' \arrow[d, "f_2"] \\
E_1 \arrow[r, "\phi"] \arrow[d, "g_1"'] & E_2 \arrow[d, "g_2"] \\
U_1 \arrow[r, "\psi"'] & U_2
\end{tikzcd}$$

such that $gf = 0$, we must show that this factors through C as $g = g'p$ for a unique $g' : C \to U$ i.e. that there exists a unique e' such that the following diagram commutes

$$\begin{tikzcd}
& E_1' \arrow[r, "\phi'"] \arrow[d, "f_1"'] & E_2' \arrow[d, "f_2"] & \\
& E_1 \arrow[r, "\phi"] \arrow[d, "p_1"] \arrow[dl, "g_1"'] & E_2 \arrow[d, "p_2"'] \arrow[dr, "g_2"] & \\
U_1 & C_1 \arrow[l, dashed, "g_1'"] \arrow[r, "\varphi"] & C_2 \arrow[r, dashed, "g_1'"] & U_2
\end{tikzcd} \tag{5.2}$$

The proof uses the universal property of the cokernel of $f_1 : E_1' \to E_1$ in $\mathbf{Coh}(X)$ and is similar to the one for the kernel so we do not go through the details.

Lemma 5.1.4. *A morphism $f : T' \to T$ given by the following commutative diagram*

$$\begin{tikzcd}
E_1' \arrow[r, "\phi'"] \arrow[d, "f_1"'] & E_2' \arrow[d, "f_2"] \\
E_1 \arrow[r, "\phi"] & E_2
\end{tikzcd}$$

is an epimorphism in $\mathcal{T}\mathbf{Coh}(X)$ if and only if f_1 and f_2 are epimorphisms in $\mathbf{Coh}(X)$.

Proof. Let $f : T' \to T$ be a morphism in $\mathcal{T}\mathbf{Coh}(X)$ and let $g, g' : T \to V$ be two morphisms where $V = (V_1, V_2, \varphi)$, so we have the following commutative diagrams

$$
\begin{tikzcd}
E_1' \arrow[r, "\phi'"] \arrow[d, "f_1"'] & E_2' \arrow[d, "f_2"] \\
E_1 \arrow[r, "\phi"] \arrow[d, "g_1"'] & E_2 \arrow[d, "g_2"] \\
V_1 \arrow[r, "\varphi"'] & V_2
\end{tikzcd}
\qquad
\begin{tikzcd}
E_1' \arrow[r, "\phi'"] \arrow[d, "f_1"'] & E_2' \arrow[d, "f_2"] \\
E_1 \arrow[r, "\phi"] \arrow[d, "g_1'"'] & E_2 \arrow[d, "g_2'"] \\
V_1 \arrow[r, "\varphi"'] & V_2
\end{tikzcd}
$$

Assume that f_1 and f_2 are epimorphisms in $\mathbf{Coh}(X)$. By definition, this implies that if $g_1 f_1 = g_1' f_1$, then $g_1 = g_1'$ and if $g_2 f_2 = g_2' f_2$, then $g_2 = g_2'$. Now if $g_1 f_1 = g_1' f_1$ and $g_2 f_2 = g_2' f_2$, then we have $gf = g'f$ in $\mathcal{T}\mathbf{Coh}(X)$. This in turn implies that $g = g'$ in $\mathcal{T}\mathbf{Coh}(X)$ (since $g_1 = g_1'$ and $g_2 = g_2'$). Hence f is an epimorphism in $\mathcal{T}\mathbf{Coh}(X)$.

Conversely, assume that f is a epimorphism in $\mathcal{T}\mathbf{Coh}(X)$. In order to show that $f_1 : E_1' \to E_1$ is an epimorphism in $\mathbf{Coh}(X)$ we start with two morphisms $g_1, g_1' : E_1 \to V_1$ such that $g_1 f_1 = g_1' f_1$. This gives us two commutative diagrams

$$
\begin{tikzcd}
E_1' \arrow[r, "\phi'"] \arrow[d, "f_1"'] & E_2' \arrow[d, "f_2"] \\
E_1 \arrow[r, "\phi"] \arrow[d, "g_1"'] & E_2 \arrow[d, "0"] \\
V_1 \arrow[r, "0"'] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
E_1' \arrow[r, "\phi'"] \arrow[d, "f_1"'] & E_2' \arrow[d, "f_2"] \\
E_1 \arrow[r, "\phi"] \arrow[d, "g_1'"'] & E_2 \arrow[d, "0"] \\
V_1 \arrow[r, "0"'] & 0
\end{tikzcd}
$$

where $g_1 f_1 = g_1' f_1$. Now since f is an epimorphism in $\mathcal{T}\mathbf{Coh}(X)$, we get that $g_1 = g_1'$ and so f_1 is an epimorphism in $\mathbf{Coh}(X)$.

Similarly we can show that $f_2 : E_2 \to E_2'$ is an epimorphism in $\mathbf{Coh}(X)$ Assume we have two morphisms $g_2, g_2' : E_2 \to V_2$ such that $g_2 f_2 = g_2' f_2$. This gives us two commutative diagrams

$$\begin{tikzcd} E_1' \arrow[r, "\phi'"] \arrow[d, "f_1"'] & E_2' \arrow[d, "f_2"] \\ E_1 \arrow[r, "\phi"] \arrow[d, "g_2\phi"'] & E_2 \arrow[d, "g_2"] \\ V_2 \arrow[r, equal] & V_2 \end{tikzcd} \qquad \begin{tikzcd} E_1' \arrow[r, "\phi'"] \arrow[d, "f_1"'] & E_2' \arrow[d, "f_2"] \\ E_1 \arrow[r, "\phi"] \arrow[d, "g_2'\phi"'] & E_2 \arrow[d, "g_2'"] \\ V_2 \arrow[r, equal] & V_2 \end{tikzcd}$$

where $g_2\phi f_1 = g_2'\phi f_1$ and $g_2 f_2 = g_2' f_2$. Now since f is an epimorphism in $\mathcal{T}\mathbf{Coh}(X)$, we get that $g_2 = g_2'$ and so f_2 is an epimorphism in $\mathbf{Coh}(X)$. $\square$

Lemma 5.1.5. *A morphism $f : T' \to T$ given by the following commutative diagram*

$$\begin{tikzcd} E_1' \arrow[r, "\phi'"] \arrow[d, "f_1"'] & E_2' \arrow[d, "f_2"] \\ E_1 \arrow[r, "\phi"] & E_2 \end{tikzcd}$$

is an monomorphism in $\mathcal{T}\mathbf{Coh}(X)$ if and only if f_1 and f_2 are monomorphisms in $\mathbf{Coh}(X)$.

Proof. The proof is similar to Lemma 5.1.4. $\square$

5.2 Stability

We have seen in Chapter 4 that the stability notion for coherent systems depends on a real parameter. The same is true for the notion of stability for holomorphic triples.

Definition 5.2.1. For any real number α, the *α-degree* of a triple $T = (E_1, E_2, \phi)$, with $\mathrm{rk}(E_1) = r_1, \mathrm{rk}(E_2) = r_2$ is

$$\deg_\alpha(T) = \deg(E_1 \oplus E_2) + \alpha r_1,$$

$$= \deg(E_1) + \deg(E_2) + \alpha r_1,$$

and the α-*slope* of T is

$$\mu_\alpha(T) = \frac{\deg_\alpha(T)}{r_1 + r_2}.$$

Definition 5.2.2. The triple $T = (E_1, E_2, \phi)$ is called α-*stable* (resp. α-*semi-stable*) if for all nontrivial subtriples (i.e. all subtiples of T except the zero triple and T itself) $T' = (E_1', E_2', \phi')$ of T we have

$$\mu_\alpha(T') < \mu_\alpha(T) \quad \text{(resp. } \leq\text{)}.$$

S. Bradlow, O. García-Prada, P. Gothen and others have studied the moduli space of stable triples (see [BG2] and [BGG]) and have shown how these moduli spaces depend on the real parameter α.

5.3 Bridgeland stability conditions

5.3.1 Is $\mathcal{T}\mathbf{Coh}(X)$ an abelian category?

We now want to see if the stability of triples fits into the framework of Bridgeland stability conditions, as outlined in Chapter 3. To begin with, we must ensure that the $\mathcal{T}\mathbf{Coh}(X)$, forms an abelian category. Let us now go through the axioms of an abelian category (Definition 1.4.27):

1. The category $\mathcal{T}\mathbf{Coh}(X)$ has a zero object, $T = 0$.

2. For every pair of objects there is a product.

Given a pair of objects $T' = (E_1', E_2', \phi')$ and $T = (E_1, E_2, \phi)$, we construct the product $P = (P_1, P_2, \varphi)$ of T' and T as follows:

$$P_1 := E_1' \oplus E_1$$

$$P_2 := E_2' \oplus E_2$$

$$\varphi := (\phi', \phi)$$

Firstly, we must show that there exists morphisms $p_1 : P \to T'$ and $p_2 : P \to T$. Consider p_1 as the following commutative diagram

$$\begin{tikzcd} P_1 \arrow[r, "\varphi"] \arrow[d, "p_{11}"'] & P_2 \arrow[d, "p_{12}"] \\ E_1' \arrow[r, "\phi'"] & E_2' \end{tikzcd}$$

Let p_{11} and p_{12} be projections to the first factor - these are sheaf morphisms. Clearly, the above diagram is commutative.

Similarly, we can define p_2 to be a morphism in $\mathcal{T}\mathbf{Coh}(X)$ given by the following commutative diagram

$$\begin{tikzcd} P_1 \arrow[r, "\varphi"] \arrow[d, "p_{21}"'] & P_2 \arrow[d, "p_{22}"] \\ E_1 \arrow[r, "\phi"] & E_2 \end{tikzcd}$$

where p_{21} and p_{22} are projections to the second factor.

Now we must show that for every pair of morphisms $X = (X_1, X_2, \psi) \to T'$ and $X \to T$, given by the following commutative diagrams

$$\begin{tikzcd} X_1 \arrow[r, "\psi"] \arrow[d] & X_2 \arrow[d] \\ E_1' \arrow[r, "\phi'"] & E_2' \end{tikzcd} \qquad \begin{tikzcd} X_1 \arrow[r, "\psi"] \arrow[d] & X_2 \arrow[d] \\ E_1 \arrow[r, "\phi"] & E_2 \end{tikzcd}$$

there is a unique $X \to P$ such that

$$\begin{tikzcd} & T' \\ X \arrow[ur] \arrow[r, dashed] \arrow[dr] & P \arrow[u] \arrow[d] \\ & T \end{tikzcd}$$

commutes, i.e. the following commutes

$$
\begin{tikzcd}
 & E_1' \arrow[r, "\phi'"] & E_2' & \\
X_1 \arrow[ur] \arrow[r, dashed] \arrow[dr] & P_1 \arrow[u, "p_{11}"'] \arrow[r] \arrow[d, "p_{21}"'] & P_2 \arrow[u, "p_{12}"] \arrow[d, "p_{22}"] & X_2 \arrow[ul] \arrow[l, dashed] \arrow[dl] \\
 & E_1 \arrow[r, "\phi"] & E_2 &
\end{tikzcd}
$$

Since the category of $\mathbf{Coh}(X)$ is abelian, we know that there exists unique morphism $X_1 \to P_1$ and $X_2 \to P_2$ making the four triangles above commutative. It remains to show that

$$
\begin{tikzcd}
X_1 \arrow[r, "\psi"] \arrow[d] & X_2 \arrow[d] \\
P_1 \arrow[r, "\varphi"] & P_2
\end{tikzcd}
$$

commutes, i.e. is a morphism in $\mathcal{T}\mathbf{Coh}(X)$. This follows from the definition of the P_2 (i.e. the product of E_2' and E_2). Since we are given morphisms $x : X_1 \to E_2$ and $y : X_1 \to E_2'$, we get a unique morphism $f : X_1 \to P_2$ so that $x = p_{12}f$ and $y = p_{22}f$. Hence the above diagram commutes.

3. Every morphism has a kernel and cokernel.

This was shown in Section 5.1.

4. Every epimorphism is the cokernel of its kernel.

Recall from Lemma 5.1.4 that a morphism $f : T' \to T$ is an epimorphism in $\mathcal{T}\mathbf{Coh}(X)$ if and only if f_1 and f_2 are epimorphisms in $\mathbf{Coh}(X)$. Assume $f : T' \to T$ is an epimorphism in $\mathcal{T}\mathbf{Coh}(X)$. We want to show that this is the cokernel of its kernel. Since $\mathbf{Coh}(X)$ is an abelian category, we know this is true here. So we have

$$
\begin{tikzcd}
E_1' \arrow[r, "\phi'"] \arrow[d, "f_1"'] & E_2' \arrow[d, "f_2"] \\
E_1 \arrow[r, "\phi"] & E_2
\end{tikzcd}
$$

with $E_1 = C_1$ (the cokernel of $K_1 \to E_1'$, where K_1 is the kernel of $f_1 : E_1' \to E_1$ in $\mathbf{Coh}(X)$). Similarly, $E_2 = C_2$ (the cokernel of $K_2 \to E_2'$, where K_2 is the kernel of $f_2 : E_2' \to E_2$).

So we have

$$\begin{tikzcd} K_1 \arrow[r] \arrow[d] & K_2 \arrow[d] \\ E_1' \arrow[r, "\phi'"] \arrow[d] & E_2' \arrow[d] \\ C_1 \arrow[r, "\phi"'] & C_2 \end{tikzcd}$$

Now from the construction of the cokernel described in Section 5.1 above, we know that ϕ is the morphism iduced by ϕ' and so we see that f is the cokernel of its kernel in $\mathcal{T}\mathbf{Coh}(X)$.

5. Every monomorphism is the kernel of its cokernel.

Recall from Lemma 5.1.5 that the morphism $f : T' \to T$ is a monomorphism in $\mathcal{T}\mathbf{Coh}(X)$ if and only if f_1 and f_2 are monomorphisms in $\mathbf{Coh}(X)$. So now let us assume that we are given $f : T' \to T$ a monomorphism in $\mathcal{T}\mathbf{Coh}(X)$. We want to show that this is the kernel of its cokernel. Since the $\mathbf{Coh}(X)$ is an abelian category, we know this is true here. So we have

$$\begin{tikzcd} E_1' \arrow[r, "\phi'"] \arrow[d, "f_1"'] & E_2' \arrow[d, "f_2"] \\ E_1 \arrow[r, "\phi"] & E_2 \end{tikzcd}$$

with $E_1' = K_1$ (the kernel of $E_1 \to C_1$, where C_1 is the cokernel of $f_1 : E_1' \to E_1$ in $\mathbf{Coh}(X)$). Similarly, $E_2' = K_2$ (the kernel of $E_2 \to C_2$, where C_2 is the cokernel of $f_2 : E_2' \to E_2$). So we have the following commutative diagram

$$\begin{tikzcd}
K_1 \arrow[r, "\phi'"] \arrow[d, "f_1"'] & K_2 \arrow[d, "f_2"] \\
E_1 \arrow[r, "\phi"] \arrow[d] & E_2 \arrow[d] \\
C_1 \arrow[r] & C_2
\end{tikzcd}$$

where ϕ' is the ϕ restircted to K_1. Hence f is the cokernel of its kernel in $\mathcal{T}\mathbf{Coh}(X)$.

Hence, all of the axioms are satisfied and so we have the following proposition.

Proposition 5.3.1. *$\mathcal{T}\mathbf{Coh}(X)$ is an abelian category.*

5.3.2 Grothendieck group

Throughout this section C will denote a smooth projective curve, unless otherwise specified. We have seen in Chapter 3, Example 3.1.2 (and Proposition 3.1.10) that the Grothendieck group of the derived category of coherent sheaves on C, $K(C) = \mathbb{Z} \oplus \mathrm{Pic}(C)$. We write $K(\mathcal{T})$ for the Grothendieck group of $\mathcal{T}\mathbf{Coh}(C)$. We begin with the following lemma that will be useful in proofs.

Remark 5.3.2. We normally write a triple $T = (E_1, E_2, \phi)$ as follows: $E_1 \to E_2$. However, for convenience, when depicting short exact sequences of triples we write a triple as follows:

$$\begin{tikzcd}
E_1 \arrow[d] \\
E_2
\end{tikzcd}$$

Note that a morphism of triples then changes accordingly. This notation will only be used in short exact sequences of triples.

Lemma 5.3.3. *Any triple $T \in \mathcal{T}\mathbf{Coh}(C)$ where $T = (E_1, E_2, \phi)$ can be placed in a short exact sequence as follows*

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & E_1 \arrow[r, equal] \arrow[d, "\phi"] & E_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & E_2 \arrow[r, equal] & E_2 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd} \tag{5.3}$$

Proof. Let $T' = (0, E_2, 0)$ and $T'' = (E_1, 0, 0)$ then we have the following

$$0 \longrightarrow T' \xrightarrow{f} T \xrightarrow{g} T'' \longrightarrow 0$$

By Definition 1.4.11, if $\mathrm{im}(f) = \ker(g)$ and if f is a monomorphism and g is an epimorphism, then (5.3) is a short exact sequence. The morphism f is given by the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r, "f_1"] \arrow[d] & E_1 \arrow[d, "\phi"] \\ E_2 \arrow[r, equal, "f_2"] & E_2 \end{tikzcd}$$

Clearly f_1 and f_2 are monomorphisms in $\mathbf{Coh}(C)$. Hence, by Lemma 5.1.5, f is a monomorphism. Similarly we can show that g is an epimorphism (using Lemma 5.1.4).

Now since every monomorphism is the kernel of its cokernel in $\mathcal{T}\mathbf{Coh}(C)$, then by definition of image we have that $\mathrm{im}(f) = T'$. It remains to show that $T' = \ker(g)$. By definition the kernel of $g : T \to T''$ is $i : K \to T$ given by the following commutative diagram

$$\begin{tikzcd} K_1 \arrow[r] \arrow[d] & E_1 \arrow[r, equal, "g_1"] \arrow[d] & E_1 \arrow[d] \\ K_2 \arrow[r] & E_2 \arrow[r, "g_2"] & 0 \end{tikzcd}$$

where K_1 and K_2 are the kernels of g_1 and g_2 in $\mathbf{Coh}(C)$. Hence clearly $K_1 = 0$ and $K_2 = E_2$. So clearly we have $T' = \ker(g)$. □

Lemma 5.3.4. *On C, a smooth projective curve, $K(\mathcal{T}) \cong K(C) \oplus K(C)$*

Proof. Each $T = (E_1, E_2, \phi) \in \mathcal{T}\mathbf{Coh}(C)$ defines an element $[T] \in K(\mathcal{T})$. We know from Lemma 5.3.3 that we can place T in a short exact sequence as follows:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & E_1 \arrow[r, equal] \arrow[d, "\phi"] & E_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & E_2 \arrow[r, equal] & E_2 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

So in $K(\mathcal{T})$, we have

$$[\, E_1 \xrightarrow{\phi} E_2 \,] = [\, 0 \longrightarrow E_2 \,] + [\, E_1 \longrightarrow 0 \,]$$

Now let us define a map

$$f : K(\mathcal{T}) \to K(C) \oplus K(C), \qquad [T] \mapsto ([E_1], [E_2])$$

Clearly this is a well-defined homomorphism of groups. We can give the inverse of f by

$$([E_1], [E_2]) \mapsto [E_1 \to 0] + [0 \to E_2].$$

Hence we have an isomorphism

$$K(\mathcal{T}) \cong K(C) \oplus K(C)$$

So we conclude

$$K(\mathcal{T}) \cong \mathbb{Z}^2 \oplus \mathrm{Pic}(C)^{\oplus 2}$$

□

Now we want to compute the numerical Grothendieck group of $\mathcal{T}\mathbf{Coh}(C)$, denoted $\mathcal{N}(\mathcal{T})$, where C is an elliptic curve. We have seen (Example 3.1.15) that the numerical Grothendieck group of the derived category of coherent sheaves on an elliptic curve C is $\mathcal{N}(C) = \mathbb{Z}^2$. Denote the bounded derived category of holomorphic triples on C by $\mathcal{D}(\mathcal{T})$. From now on C, will denote an elliptic curve. We begin with some lemmas.

Lemma 5.3.5. *(i)* $\mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(0 \to E_2, E_1[i] \to 0) = 0$ *for all* $i \in \mathbb{Z}$.

(ii) $\mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(E_1 \to 0, 0 \to E_2) = 0$.

(iii) $\mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(E_1 = E_1, 0 \to E_2[i]) = 0$ *for all* $i \in \mathbb{Z}$.

Proof. For the purpose of this proof we will depict the triples vertically. We know morphisms in $\mathcal{T}\mathbf{Coh}(C)$ are roofs (see Definition 1.4.47 and Remark 1.4.48).

(i) Let the roof

$$\begin{tikzcd} 0 \arrow[d] & F_1^\bullet \arrow[l, "s_1"'] \arrow[r, "f_1"] \arrow[d, "\varphi"] & E_1[i] \arrow[d] \\ E_2 & F_2^\bullet \arrow[l, "s_2"] \arrow[r, "f_2"'] & 0 \end{tikzcd} \tag{5.4}$$

represent a homomorphism in $\mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(0 \to E_2, E_1[i] \to 0)$. Here s_1 and s_2 are quasi-isomorphisms. The same morphism is represented by any roof obtained by composing the above with quasi-isomorphisms h_1, h_2:

$$\begin{tikzcd} \tilde{F_1}^\bullet \arrow[r, "h_1"] \arrow[d] & F_1^\bullet \arrow[d, "\varphi"] \\ \tilde{F_2}^\bullet \arrow[r, "h_2"] & F_2^\bullet \end{tikzcd}$$

Because $s_1 : F_1^\bullet \to 0$ is a quasi-isomorphism, we can choose $\tilde{F_1}^\bullet = 0, \tilde{F_2}^\bullet = F_2^\bullet$ and obtain a commutative diagram whose horizontal arrows are quasi-isomorphisms:

$$\begin{tikzcd} 0 \arrow[r, "h_1"] \arrow[d] & F_1^\bullet \arrow[d, "\varphi"] \\ F_2^\bullet \arrow[r, equal, "h_2"] & F_2^\bullet \end{tikzcd}$$

Therefore, the above morphism (5.4) can also be represented by

$$\begin{tikzcd} 0 \arrow[d] & 0 \arrow[l, "0"'] \arrow[r, "0"] \arrow[d] & E_1[i] \arrow[d] \\ E_2 & F_2^\bullet \arrow[l, "s_2"] \arrow[r, "f_2=0"'] & 0 \end{tikzcd} \tag{5.5}$$

where $h_1 \circ s_1 = 0, h_1 \circ f_1 = 0, h_2 \circ s_2 = s_2, h_2 \circ f_2 = f_2 = 0$. Hence the morphism (5.5) is zero.

(ii) This is clear because by Remark 1.4.49

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(T, T') = \mathrm{Hom}_{\mathcal{T}\mathbf{Coh}(C))}(T, T')$$

for objects T, T' of $\mathcal{T}\mathbf{Coh}(C)$).

(iii) Let the roof

$$\begin{tikzcd} E_1 \arrow[d, equal] & F_1^\bullet \arrow[l, "s_1"'] \arrow[r, "f_1"] \arrow[d, "\varphi"] & 0 \arrow[d] \\ E_1 & F_2^\bullet \arrow[l, "s_2"] \arrow[r, "f_2"'] & 0 \end{tikzcd}$$

represent a homomorphism in $\mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(E_1 = E_1, 0 \to E_2[i])$. Again, s_1 and s_2 are quasi-isomorphisms. This implies that φ is a quasi-isomorphism as well. Hence we can compose with

$$\begin{tikzcd} F_1^\bullet \arrow[r, equal] \arrow[d, equal] & F_1^\bullet \arrow[d, "\varphi"] \\ F_1^\bullet \arrow[r, "\varphi"] & F_2^\bullet \end{tikzcd}$$

so that the morphism above is also represented by

$$\begin{tikzcd} E_1 \arrow[d, equal] & F_1^\bullet \arrow[l, "s_1"'] \arrow[r, "f_1=0"] \arrow[d, equal] & 0 \arrow[d] \\ E_1 & F_1^\bullet \arrow[l, "s_2\circ\varphi"] \arrow[r, "f_2\circ\varphi"'] & E_2[i] \end{tikzcd}$$

The commumativity of the right square gives $f_2 \circ \varphi = 0$, hence this roof represents the zero morphism. ☐

Lemma 5.3.6. *If $E_1, E_1 \in \mathbf{Coh}(C), i \in \mathbb{Z}$ then*

$$\mathrm{Hom}_{\mathcal{D}(C)}(E_1, E_2[i]) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(E_1 \to 0, 0 \to E_2[i+1]).$$

In particular, if C is a smooth curve,

$$\mathrm{Hom}_{\mathcal{D}(C)}(E_1, E_2) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(E_1 \to 0, 0 \to E_2[1]).$$

and $\mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(E_1 \to 0, 0 \to E_2[i]) = 0$ *for all $i \neq 1, 2$ and*

$$\mathrm{Ext}^1_{\mathbf{Coh}(C)}(E_1, E_2) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(E_1 \to 0, 0 \to E_2[2]).$$

Proof. Applyling the contravariant functor $\mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(-,T)$, where the triple $T=(0,E_2,0)$, to the distinguished triangle in $\mathcal{D}(\mathcal{T})$ which corresponds to the short exact sequence in $\mathcal{T}\mathbf{Coh}(C)$

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & E_1 \arrow[r, equal] \arrow[d, equal] & E_1 \arrow[r, equal] \arrow[d] & 0 \\ 0 \arrow[r] & E_1 \arrow[r, equal] & E_1 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

we obtain exact sequences for all $i \in \mathbb{Z}$:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(E_1 = E_1, T[i]) &\longrightarrow \mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(0 \to E_1, T[i]) \longrightarrow \\ \mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(E_1 \to 0, T[i+1]) &\longrightarrow \mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(E_1 = E_1, T[i+1]) \longrightarrow \end{aligned}$$

where $\mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(E_1 = E_1, T[i]) = 0$ and $\mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(E_1 = E_1, T[i+1]) = 0$ by Lemma 5.3.5. Hence we have

$$\mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(0 \to E_1, 0 \to E_2[i]) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{T})}(E_1 \to 0, 0 \to E_2[i+1])$$

□

The Euler form on $\mathcal{T}\mathbf{Coh}(C)$ is

$$\chi(T,T') = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}}(T, T'[i]),$$

where $T, T' \in \mathcal{T}\mathbf{Coh}(C)$ and $\mathcal{D} = \mathcal{D}^b(\mathcal{T}\mathbf{Coh}(C))$.

Proposition 5.3.7. *Sending $[E_1 \to E_2]$ to (r_1, d_1, r_2, d_2) gives an isomorphism $\mathcal{N}(\mathcal{T}) \xrightarrow{\sim} \mathbb{Z}^4$, where $r_i = \mathrm{rk}(E_i), d_i = \deg(E_i)$ for $i = 1, 2$.*

Proof. We can place $T = (E_1, E_2, \phi)$ in a short exact sequence as follows (Lemma 5.3.3):

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & E_1 \arrow[r, equal] \arrow[d, "\phi"] & E_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & E_2 \arrow[r, equal] & E_2 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

Define functors $I_1 : \mathbf{Coh}(C) \to \mathcal{T}\mathbf{Coh}(C)$ given by $I_1(E_1) = E_1 \to 0$ and $I_2 : \mathbf{Coh}(C) \to \mathcal{T}\mathbf{Coh}(C)$ given by $I_2(E_2) = 0 \to E_2$. Since $\chi(-,T')$ is an additive function on exact sequences, from the short exact sequence above, we get

$$\chi(T,T') = \chi(I_2(E_2),T') + \chi(I_1(E_1),T')$$

We can also place $T' = (E_1', E_2', \phi')$ in a short exact sequence as follows:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & E_1' \arrow[r, equal] \arrow[d, "\phi'"] & E_1' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E_2' \arrow[r, equal] & E_2' \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

So we get

$$\chi(T,T') = \chi(I_2(E_2), I_1(E_1')) + \chi(I_2(E_2), I_2(E_2'))$$

$$+\chi(I_1(E_1), I_1(E_1')) + \chi(I_1(E_1), I_2(E_2')).$$

From Lemma 5.3.5 (i), we have that $\mathrm{Hom}(0 \to E_2, E_1'[i] \to 0) = 0$ for all $i \in \mathbb{Z}$, hence $\chi(I_2(E_2), I_1(E_1')) = 0$. So now we have

$$\chi(T,T') = \chi(I_2(E_2), I_2(E_2')) + \chi(I_1(E_1), I_1(E_1')) + \chi(I_1(E_1), I_2(E_2')).$$

By defintion we know

$$\chi(I_1(E_1), I_2(E_2')) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}}(I_1(E_1), I_2(E_2')[i]).$$

Then by Lemma 5.3.6, we have

$$\chi(I_1(E_1), I_2(E_2')) = \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}(C)}(E_1, E_2'[i-1])$$

$$= -\sum_i (-1)^{i-1} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{D}(C)}(E_1, E_2'[i-1]) = -\chi(E_1, E_2),$$

where $\mathcal{D}(C)$ denotes the bounded derived category of coherent sheaves on C. From this we get

$$\chi(T,T') = \chi(E_2, E_2') + \chi(E_1, E_1') - \chi(E_1, E_2').$$

By Riemann-Roch (see again Example 3.1.15), we then get

$$\chi(T,T') = r_2d_2' - r_2'd_2 + r_1d_1' - r_1'd_1 - r_1d_2' + r_2'd_1$$

where $r_i := \mathrm{rk}(E_i), d_i := \deg(E_i), r_i' := \mathrm{rk}(E_i')$ and $d_i' := \deg(E_i')$ for $i = 1, 2$ and where $E_i, E_i' \in \mathbf{Coh}(C)$.

Now by definition $T \in K(\mathcal{T})^\perp$ if and only if $\chi(T,T') = 0$ for all $T' \in K(\mathcal{T})$. We can see that $\chi(T,T') = 0$ for all T' if and only if $r_1 = 0, d_1 = 0, r_2 = 0$ and $d_2 = 0$. This means that the kernel of $K(\mathcal{T}) \to \mathbb{Z}^4$, given by $[T] \mapsto (\mathrm{rk}(E_1), \deg(E_1), \mathrm{rk}(E_2), \deg(E_2))$, where $T = (E_1, E_2, \phi)$ is $K(\mathcal{T})^\perp$. Hence we get a short exact sequence

$$0 \to K(\mathcal{T})^\perp \to K(\mathcal{T}) \to \mathbb{Z}^4 \to 0$$

and so $K(\mathcal{T})/K(\mathcal{T})^\perp \cong \mathbb{Z}^4$, i.e. $\mathcal{N}(\mathcal{T}) \cong \mathbb{Z}^4$. □

Note that the result of the calculation

$$\chi(T,T') = r_2d_2' - r_2'd_2 + r_1d_1' - r_1'd_1 - r_1d_2' + r_2'd_1$$

above can also be found in [BGG] Proposition 3.2.

5.3.3 Stability conditions

The functors $I_\nu : \mathbf{Coh}(C) \to \mathcal{T}\mathbf{Coh}(C)$ for $\nu = 1, 2$ induce homomorphisms of groups

$$i_\nu : K(C) \to K(\mathcal{T}) \cong K(C) \oplus K(C)$$

where $i_1([E]) = [E \to 0]$ and $i_2([E]) = [0 \to E]$. In terms of identifications $K(C) \cong \mathbb{Z}^2, [E] \mapsto (\mathrm{rk}(E), \deg(E))$ and $K(\mathcal{T}) \cong \mathbb{Z}^4, [E_1 \to E_2] \mapsto (\mathrm{rk}(E_1), \deg(E_1), \mathrm{rk}(E_2), \deg(E_2))$, these are just the inclusions

$$i_1 : \mathbb{Z}^2 \hookrightarrow \mathbb{Z}^4 \quad i_1(a,b) = (a,b,0,0)$$

and

$$i_2 : \mathbb{Z}^2 \hookrightarrow \mathbb{Z}^4 \quad i_2(a,b) = (0,0,a,b).$$

If $Z : K(\mathcal{T}) \to \mathbb{C}$ is a stability function, we denote $Z_\nu := Z \circ i_\nu : K(C) \to \mathbb{C}$.

Lemma 5.3.8. *Let $Z : K(\mathcal{T}) \to \mathbb{C}$ be a stability function on the abelian category $\mathcal{T}\mathbf{Coh}(C)$.*

(i) Z_1 and Z_2 are stability functions on $\mathbf{Coh}(C)$.

(ii) If Z has the Harder-Narasimhan property on $\mathcal{T}\mathbf{Coh}(C)$ then Z_1 and Z_2 have the Harder-Narasimhan property on $\mathbf{Coh}(C)$.

Proof. (i) This is clear, because $Z_1([E]) = Z([E \to 0])$ and $Z_2([E]) = Z([0 \to E])$ and both are in the strict upper half-plane.

(ii) If E is a coherent sheaf on C, by assumption the triple $E \to 0$ has a Harder-Narasimhan filtration (with respect to Z). Because the only subobject of 0 is 0, the objects of this filtration are triples of the form $E_i \to 0$. Now $Z([E_i/E_{i-1} \to 0]) = Z_1(E_i/E_{i-1}])$, hence it follows that the E_i form a Harder-Narasimhan filtration for E. The proof for Z_2 is similar. $\square$

Lemma 5.3.9. *If $Z : K(\mathcal{T}) \to \mathbb{C}$ is a stability function with Harder-Narasimhan property such that the induced slicing on $\mathcal{T}\mathbf{Coh}(C)$ is locally finite, then (using the identification $K(\mathcal{T}) \cong \mathbb{Z}^4$ introduced above):*

$$Z(r_1, d_1, r_2, d_2) = -A_1 d_1 - A_2 d_2 + B_1 r_1 + B_2 r_2 + i(C_1 r_1 + C_2 r_2)$$

with real numbers A_i, B_i, C_i satisfying $A_1 > 0, A_2 > 0, C_1 > 0$ and $C_2 > 0$.

Proof. Because $Z_\nu : \mathbb{Z}^2 \to \mathbb{C}$ is a homomorphism of groups, there exists real numbers $A_\nu, B_\nu, C_\nu, D_\nu$ such that $Z_\nu(r, d) = -A_\nu d + B_\nu r + i(C_\nu r + D_\nu d)$. If $P_0 \in X$ is a point and $n \in \mathbb{Z}$, the line bundle $\mathcal{O}(nP_0)$ has degree n. Since Z_ν is a stability function, then $Z_\nu([\mathcal{O}(nP_0)]) = Z_\nu(1, n)$ is in the strict upper half-plane, i.e. $C_\nu + nD_\nu \geq 0$ for all $n \in \mathbb{Z}$. Hence, $D_\nu = 0$ and $C_\nu \geq 0$.

On the other hand, the torsion sheaf $\mathbb{C}_{P_0}$, which sits in an exact sequence

$$0 \to \mathcal{O}(-P_0) \to \mathcal{O} \to \mathbb{C}_{P_0} \to 0,$$

has rank 0 and degree 1. Thus, $Z_\nu(0, 1) = -A_\nu$ has to be in the strict upper half-plane as well. This implies $A_\nu > 0$.

If we had $C_\nu = 0$, for each $E \in \mathbf{Coh}(C)$ we obtained $Z([E \to 0]) = Z_1([E]) \in \mathbb{R}$ (if $\nu = 1$) or $Z([0 \to E]) = Z_2([E]) \in \mathbb{R}$ (if $\nu = 2$). If $\mathcal{P}$

denotes the slicing induced by Z on $\mathcal{T}\mathbf{Coh}(C)$, we obtain $\mathbf{Coh}(C) \subset \mathcal{P}(1)$. But $\mathbf{Coh}(C)$ is not of finite length, hence $\mathcal{P}$ would not be locally finite. Therefore, we must have $C_\nu > 0$. $\square$

Proposition 5.3.10. *If A_i, B_i, C_i (for $i = 1, 2$) are real numbers such that $A_1 > 0, A_2 > 0, C_1 > 0$ and $C_2 > 0$, then*

$$Z(r_1, d_1, r_2, d_2) := -A_1 d_1 - A_2 d_2 + B_1 r_1 + B_2 r_2 + i(C_1 r_1 + C_2 r_2)$$

is a stability function on $\mathcal{T}\mathbf{Coh}(C)$ which has the Harder-Narasimhan property and the corresponding slicing is locally finite.

Proof. Let $E_1 \to E_2$ be an object of $\mathcal{T}\mathbf{Coh}(C)$ and let $r_i = \mathrm{rk}(E_i)$ and $d_i = \deg(E_i)$ for $i = 1, 2$. We know $r_1 \geq 0$ and $r_2 \geq 0$, hence $C_1 r_1 + C_2 r_2 \geq 0$ and this expression can only be zero if $r_1 = r_2 = 0$. In this case, E_1 and E_2 are torsion sheaves, hence $d_1 \geq 0$ and $d_2 \geq 0$ and $Z([E_1 \to E_2]) = -A_1 d_1 - A_2 d_2 \leq 0$. This can only be zero if $d_1 = d_2 = 0$, which happens if $E_1 = E_2 = 0$. This shows that Z is in fact a stability function.

In order to prove the Harder-Narasimhan property, we are going to apply Proposition 3.1.8. We first show that if $E_1 \to E_2$ is a given triple, then the set of real numbers

$$\{\phi(F_1 \to F_2) | F_1 \to F_2 \text{ is a subtriple of } E_1 \to E_2 \text{ and}$$

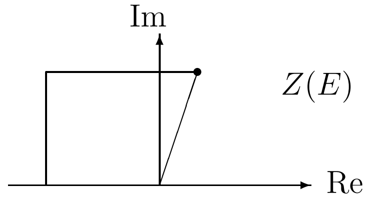
$$\phi(F_1 \to F_2) > \phi(E_1 \to E_2)\}$$

is finite. Let us denote $r_i^0 := \mathrm{rk}(E_i), d_i^0 := \deg(E_i)$ and $r_i := \mathrm{rk}(F_i), d_i := \deg(F_i)$ for $i = 1, 2$. If $F_1 \to F_2$ is a subtriple of $E_1 \to E_2$, then $0 \leq r_i \leq r_i^0$ and $r_i \in \mathbb{Z}$. Hence, for subtriples of the fixed triple $E_1 \to E_2$, the expression

$$B_1 r_1 + B_2 r_2 + i(C_1 r_1 + C_2 r_2)$$

takes only finitely many values. In particular, there exist real numbers $b < B$ and $0 \leq c < C$ such that $b \leq B_1 r_1 + B_2 r_2 \leq B$ and $c \leq C_1 r_1 + C_2 r_2 \leq C$ for all subtriples of $E_1 \to E_2$.

On the other hand, the degrees of all subsheaves of E_i is bounded above (as in Lemma 2.1.32), i.e. there exist integers D_ν with $d_\nu \leq D_\nu$ for all subtriples $F_1 \to F_2$ of $E_1 \to E_2$. Hence, the expression $-A_1 d_1 - A_2 d_2$ is bounded below. Hence $\mathrm{Re}(Z([F_1 \to F_2]))$ (i.e. the real part of the complex number) is bounded below for subtriples of $E_1 \to E_2$. This means that $Z([F_1 \to F_2])$ is in the enclosed trapezoid area of the diagram below for all subtriples $F_1 \to F_2$ of $E_1 \to E_2$ for which we have $\phi(F_1 \to F_2) > \phi(E_1 \to E_2)$



In particular, there exists a real number M such that $\mathrm{Re}(Z([F_1 \to F_2])) < M$ for all subtriples $F_1 \to F_2$ as above. Hence, $-A_1 d_1 - A_2 d_2 < M - b$ and so

$$-A_1 d_1 < M - b + A_2 d_2 \leq M - b + A_2 D_2$$

as well as

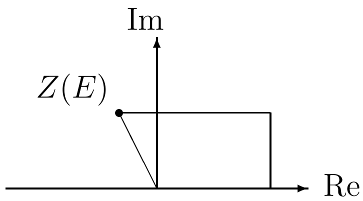
$$-A_2 d_2 < M - b + A_1 d_1 \leq M - b + A_1 D_1.$$

Because $A_\nu > 0$ this shows that the integers d_1 and d_2 are bounded below. Therefore, the expression $-A_1 d_1 - A_2 d_2$ takes only finitely many values for those subtriples $F_1 \to F_2$ of $E_1 \to E_2$ for which we have $\phi(F_1 \to F_2) > \phi(E_1 \to E_2)$. Hence, Z takes only finitely many values on this set and so the number of different slopes is finite as well. This shows that there does not exist an infinite sequence of subobjects $E_1 \to E_2$ in $\mathcal{T}\mathbf{Coh}(C)$ with strictly increasing slope.

The proof that there does not exist an infinite sequence of epimorphisms in $\mathcal{T}\mathbf{Coh}(C)$ with strictly decreasing slopes is similar. The main difference is that there exists lower bounds for quotients of a given coherent sheaf, which follows from the exact sequence

$$0 \to K \to E \to \mathcal{Q} \to 0$$

and the additivity of the degree. The picture looks now like this



The proof of the local finiteness of the corresponding slicing uses a similar argument. □

Corollary 5.3.11. *If* $\mathrm{Stab}^0(\mathcal{D})$ *denotes the set of locally finite stability conditions on* $\mathcal{D} = \mathcal{D}^b(\mathcal{T}\mathbf{Coh}(C))$ *whose heart is* $\mathcal{T}\mathbf{Coh}(C)$ *and if* $\mathrm{Stab}^0(C)$ *denotes the set of locally finite stability conditions on* $\mathcal{D}^b(\mathbf{Coh}(C))$ *whose heart is* $\mathbf{Coh}(C)$*, then*

$$\mathrm{Stab}^0(\mathcal{D}) = \mathrm{Stab}^0(C) \times \mathrm{Stab}^0(C)$$

and this identification is given by $Z \mapsto (Z_1, Z_2)$*.*

Proof. Recall (Example 3.4.1) that $\tilde{\mathrm{GL}}^+(2,\mathbb{R})$ acts transitively on $\mathrm{Stab}(C)$, the set of all locally finite stability conditions. From this, it follows easily that the stability functions which describe the elements of $\mathrm{Stab}^0(C)$ are precisely those of the form $-Ad + Br + iCr$ with $A > 0, C > 0$. The statement now follows from the previous proposition. □

So we have taken the first steps to describing the space of stability conditions on the category $\mathcal{T}\mathbf{Coh}(C)$. We first showed that $\mathcal{T}\mathbf{Coh}(X)$ is indeed an abelian category, where X is a compact Riemann surface. We then described the Grothendieck group and the numerical Grothedieck group of this category. From here, we studied the stability functions on $\mathcal{T}\mathbf{Coh}(C)$ (the category of holomorphic triples on an elliptic curve C) and finally we showed that

$$\mathrm{Stab}^0(\mathcal{D}) = \mathrm{Stab}^0(C) \times \mathrm{Stab}^0(C).$$

However, many interesting questions still remain open. Firstly, we saw that a stability function Z on $\mathcal{T}\mathbf{Coh}(C)$ with the Harder-Narasimhan property is given by

$$Z(r_1, d_1, r_2, d_2) := -A_1d_1 - A_2d_2 + B_1r_1 + B_2r_2 + i(C_1r_1 + C_2r_2).$$

Denote Z^α to be the stability function where $A_1 = A_2 = 1, B_1 = \alpha, B_2 = 0, C_1 = C_2 = 1$, i.e.

$$Z^\alpha(r_1, d_1, r_2, d_2) := -d_1 - d_2 - \alpha r_1 + i(r_1 + r_2).$$

The moduli space corresponding to this stability function was studied in [BGG]. However, it would be interesting to see whether the other stability functions, not of this form, give new moduli spaces.

Another problem that remains is to give a complete description for the set, $\mathrm{Stab}(\mathcal{D})$, of locally finite stability conditions on $\mathcal{D} = \mathcal{D}^b(\mathcal{T}\mathbf{Coh}(C))$. For example, would it be true that

$$\mathrm{Stab}(\mathcal{D}) = \mathrm{Stab}(C) \times \mathrm{Stab}(C)?$$

To answer this question would require a better understanding of more general t-structures on $\mathcal{D}$. It would also be interesting to study the relationship between $\mathrm{Stab}(\mathcal{D})$ and stability conditions on coherent systems using the set of locally finite stability conditions on the derived category of the category $\mathcal{C}(C)$ (as described at the end of Chapter 4).

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